Supplemental Appendix for
"Bootstrapping Density-Weighted Average Derivatives"
(Intended for web-posting only.)

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This Supplemental Appendix is self-contained, and the notation within this document is not always identical to the one employed in the main paper. All the results presented in the main paper are stated and proved in this document.

1 Notation

We assume throughout that $Z_n = \{z_i = (y_i, x_i')' : i = 1, \ldots, n\}$ is a random sample of the random vector $z = (y, x')'$ where $y \in R$ and $x \in R^d$ with density $f(x)$. The parameter of interest is $\theta = \mathbb{E}[f(x)\, \partial g(x)/\partial x]$ with $g(x) = \mathbb{E}[y|\, x]$. In addition, define $e(x) = f(x)\, g(x)$ and $v(x) = \mathbb{E}\left[y^2 \middle|\, x\right]$.

1.1 Sample

Define,

$$\hat{\theta}_n(h) = \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} U(z_i, z_j; h), \qquad U(z_i, z_j; h) = -h^{-(d+1)} \dot{K}\left(\frac{x_i - x_j}{h}\right)(y_i - y_j),$$

$$\hat{L}_n(z_i; h) = 2\left[(n-1)^{-1} \sum_{j=1}^{n} U(z_i, z_j; h) - \hat{\theta}_n(h)\right],$$

$$\hat{W}_n(z_i, z_j; h) = U(z_i, z_j; h) - \frac{1}{2}\left(\hat{L}_n(z_i; h) + \hat{L}_n(z_j; h)\right) - \hat{\theta}_n(h),$$

along with

$$\hat{\Sigma}_n(h) = n^{-1} \sum_{i=1}^{n} \hat{L}_n(z_i; h)\, \hat{L}_n(z_i; h)', \quad \hat{\Delta}_n(h) = h^{d+2} \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \hat{W}_n(z_i, z_j; h)\, \hat{W}_n(z_i, z_j; h)',$$

$$\hat{\Delta}_{2,n}(h) = h^{d+2}\left[\binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} U(z_i, z_j; h)\, U(z_i, z_j; h)' - \hat{\theta}_n(h)\, \hat{\theta}_n(h)'\right],$$

$$\hat{\Delta}_{3,n}(h) = h^{d+2} \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} U(z_i, z_j; h)\, U(z_i, z_j; h)'.$$

Note, the definition $\hat{L}_n(z_i; h)$ is consistent with that of Cattaneo, Crump and Jansson (2011) since $U(z_i, z_j; h) = 0$ for $i = j$. Now, the Hoeffding decomposition yields,

$$\hat{\theta}_n(h) = \theta(h) + n^{-1} \sum_{i=1}^{n} L(z_i; h) + \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} W(z_i, z_j; h),$$

where

$$\theta(h) = \mathbb{E}[U(z_i, z_j; h)], \qquad L(z_i; h) = n\left[\mathbb{E}\left[\hat{\theta}_n(h) \middle|\, z_i\right] - \theta(h)\right] = 2\left[\mathbb{E}[U(z_i, z_j; h)|\, z_i] - \theta(h)\right],$$

$$\begin{aligned} W(z_i, z_j; h) &= \binom{n}{2}\left[U(z_i, z_j; h) - \mathbb{E}\left[\hat{\theta}_n(h)\middle| z_i\right] - \mathbb{E}\left[\hat{\theta}_n(h)\middle| z_j\right] + \theta(h)\right] \\ &= U(z_i, z_j; h) - \frac{1}{2}\left(L(z_i; h) + L(z_j; h)\right) - \theta(h). \end{aligned}$$

The variance decomposition is

$$\mathbb{V}[\hat{\theta}_n(h)] = \frac{1}{n}\mathbb{V}\left[L(z_i; h)\right] + \binom{n}{2}^{-1}\mathbb{V}\left[W(z_i, z_j; h)\right].$$

1.2 Bootstrap Sample

Let $Z_n^* = \left\{z_i^* = (y_i^*, x_i^{*\prime})' : i = 1, \ldots, m(n)\right\}$ be a random sample with replacement from the observed sample Z_n. A "$*$" superscript will denote expectation with respect to the empirical measure (conditional on the observed sample, Z_n). Then the bootstrap analogs of the above estimators are,

$$\hat{\theta}_m^*(h) = \binom{m}{2}^{-1}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} U\left(z_i^*, z_j^*; h\right), \qquad U\left(z_i^*, z_j^*; h\right) = -h^{-(d+1)}\dot{K}\left(\frac{x_i^* - x_j^*}{h}\right)\left(y_i^* - y_j^*\right),$$

$$\hat{L}_m^*(z_i^*; h) = 2\left[(m-1)^{-1}\sum_{j=1}^{m} U\left(z_i^*, z_j^*; h\right) - \hat{\theta}_m^*(h)\right],$$

$$\hat{W}_m^*\left(z_i^*, z_j^*; h\right) = U\left(z_i^*, z_j^*; h\right) - \frac{1}{2}\left(\hat{L}_m^*(z_i^*; h) + \hat{L}_m^*\left(z_j^*; h\right)\right) - \hat{\theta}_m^*(h),$$

along with,

$$\hat{\Sigma}_m^*(h) = m^{-1}\sum_{i=1}^{m}\hat{L}_m^*(z_i^*; h)\,\hat{L}_m^*(z_i^*; h)', \quad \hat{\Delta}_m^*(h) = h^{d+2}\binom{m}{2}^{-1}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\hat{W}_m^*\left(z_i^*, z_j^*; h\right)\hat{W}_m^*\left(z_i^*, z_j^*; h\right)',$$

$$\hat{\Delta}_{2,m}^*(h) = h^{d+2}\left[\binom{m}{2}^{-1}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} U\left(z_i^*, z_j^*; h\right)U\left(z_i^*, z_j^*; h\right)' - \hat{\theta}_m^*(h)\,\hat{\theta}_m^*(h)'\right],$$

$$\hat{\Delta}_{3,m}^*(h) = h^{d+2}\binom{m}{2}^{-1}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} U\left(z_i^*, z_j^*; h\right)U\left(z_i^*, z_j^*; h\right)'.$$

The (conditional) Hoeffding decomposition is,

$$\hat{\theta}_m^*(h) = \theta^*(h) + m^{-1}\sum_{i=1}^{m} L^*(z_i^*; h) + \binom{m}{2}^{-1}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} W^*\left(z_i^*, z_j^*; h\right),$$

where

$$\theta^*(h) = \mathbb{E}^*\left[U\left(z_i^*, z_j^*; h\right)\right], \qquad L^*(z_i^*; h) = 2\left[\mathbb{E}^*\left[U\left(z_i^*, z_j^*; h\right)\middle| z_i^*\right] - \theta^*(h)\right],$$

$$W^*\left(z_i^*, z_j^*; h\right) = U\left(z_i^*, z_j^*; h\right) - \frac{1}{2}\left(L^*(z_i^*; h) + L^*\left(z_j^*; h\right)\right) - \theta^*(h).$$

1.3 Other Notation

Throughout the Appendix let C denote a generic constant satisfying $0 < C < \infty$. Define $\Lambda = \left\{\lambda \in \mathbb{R}^d : \|\lambda\| = 1\right\}$ where $\|\cdot\|$ denotes the Euclidean norm. When there is no risk of confusion we will write $U_{h,ij}$ and $U^*_{h,ij}$ for $U(z_i, z_j; h)$ and $U\left(z_i^*, z_j^*; h\right)$, respectively.

2 Preliminary Lemmas

The following lemmas will be used repeatedly throughout the Appendix. The proofs are given after the statement of the lemmas. Let $\dot{f}$ and $\ddot{f}$ denote the first and second derivatives of $f(x)$ (and similarly for other functions).

Lemma 2.1: Let $\mathbb{E}|y_i|^\sigma < \infty$ for $\sigma \in N$ and assume f is bounded and $\dot{K}$ is bounded and integrable. For $\sigma \geq 2$,

$$\mathbb{E}\left[\left|\lambda' U_{h,ij}\right|^\sigma\right] \leq C h^{-(\sigma-1)d-\sigma}.$$

If $\sigma = 1$, additionally assume that vf is bounded. Then,

$$\mathbb{E}\left[\lambda' U_{h,ij}\right] \leq C.$$

Lemma 2.2: Let $\mathbb{E}|y_i|^\sigma < \infty$ and assume that f, e, $\dot{e}$ and vf are bounded, and $\dot{K}$ is bounded and integrable. For $\sigma = 2$,

$$\mathbb{E}\left[\left|\lambda' U_{h,ij}\right|^2 \middle| z_i\right] \leq C\left(|y_i|^2 + 1\right) h^{-d-2},$$

and

$$\mathbb{E}\left[\lambda' U_{h,ij} \middle| z_i\right] \leq C\left(|y_i| + 1\right).$$

Lemma 2.3: Assume that f, $\dot{f}$, $\ddot{f}$, e, $\dot{e}$ and vf are bounded, and $\dot{K}$ is bounded and integrable. Then,

$$\mathbb{E}\left[\left(\lambda' U_{h,ij}\right) \mathbb{E}\left[\lambda' U_{h,jk} \middle| z_j\right] \middle| z_i\right] \leq C\left(|y_i| + 1\right).$$

Lemma 2.4: Let Assumptions M and K in the main paper hold. Then,

$$\mathbb{E}\left[\left(\mathbb{E}\left[\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right) \middle| z_i, z_j\right]\right)^2\right] \leq C h^{-d-4}.$$

Proof of Lemma 2.1: This result follows by the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left|\lambda' U_{h,ij}\right|^\sigma\right] \leq \|\lambda\|^\sigma \mathbb{E}\left[\|U_{h,ij}\|^\sigma\right] = \mathbb{E}\left[\|U_{h,ij}\|^\sigma\right],$$

and by the proof of Lemma 4 in Robinson (1995) when $\sigma > 1$ and by Lemma 2.2. when $\sigma = 1$.

Proof of Lemma 2.2: This result follows by the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left|\lambda' U_{h,ij}\right|^2 \middle| z_i\right] \leq \|\lambda\|^2 \mathbb{E}\left[\left\|U_{h,ij}\right\|^2 \middle| z_i\right] = \mathbb{E}\left[\left\|U_{h,ij}\right\|^2 \middle| z_i\right],$$

and by the proof of Lemma 5 in Robinson (1995) for the case of $\sigma = 2$. For the case of $\sigma = 1$, by integration by parts,

$$\mathbb{E}\left[\lambda' U_{h,ij} | z_i\right] = -y_i \int \lambda' \dot{f}\left(x_i - uh\right) K\left(u\right) du + \int \lambda' \dot{e}\left(x_i - uh\right) K\left(u\right) du,$$

and so,

$$\begin{aligned}
\left|\mathbb{E}\left[\lambda' U_{h,ij} | z_i\right]\right| &= \left| y_i \int \lambda' \dot{f}\left(x_i - uh\right) K\left(u\right) du - \int \lambda' \dot{e}\left(x_i - uh\right) K\left(u\right) du \right| \\
&\leq |y_i| \int \left|\lambda' \dot{f}\left(x_i - uh\right)\right| K\left(u\right) du + \int \left|\lambda' \dot{e}\left(x_i - uh\right)\right| K\left(u\right) du \\
&= |y_i| \int \left\|\dot{f}\left(x_i - uh\right)\right\| K\left(u\right) du + \int \left\|\dot{e}\left(x_i - uh\right)\right\| K\left(u\right) du \\
&\leq C\left(|y_i| + 1\right).
\end{aligned}$$

Proof of Lemma 2.3: See Lemma 3 in Robinson (1995).

Proof of Lemma 2.4: The result follows as in Lemma 6 in Nishiyama and Robinson (2000).

3 Non-bootstrap Statistics

3.1 Expansions and Convergence in Probability of $n-$varying U-statistics

We will begin by investigating the properties of $\hat{\Sigma}_n(h)$, $\hat{\Delta}_n(h)$, $\hat{\Delta}_{2,n}(h)$ and $\hat{\Delta}_{3,n}(h)$. The following n-varying U-statistics serve as the building blocks of these estimators. For $\lambda \in \Lambda$ define,

$$\tilde{T}_{1,n}^{(s)}\left(\lambda; h\right) = \binom{n}{2}^{-1} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\lambda' U_{h,ij}\right)^s,$$

$$\begin{aligned}
\tilde{T}_{2,n}^{(s)}\left(\lambda; h\right) &= \binom{n}{3}^{-1} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} U_{h,ijk}^{(s)}\left(\lambda\right), \\
& \qquad U_{h,ijk}^{(s)}\left(\lambda\right) = \frac{\left(\lambda' U_{h,ij}\right)^s \left(\lambda' U_{h,ik}\right)^s + \left(\lambda' U_{h,ij}\right)^s \left(\lambda' U_{h,jk}\right)^s + \left(\lambda' U_{h,ik}\right)^s \left(\lambda' U_{h,jk}\right)^s}{3},
\end{aligned}$$

$$\begin{aligned}
\tilde{T}_{3,n}^{(s)}\left(\lambda; h\right) &= \binom{n}{4}^{-1} \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} U_{h,ijkl}^{(s)}\left(\lambda\right), \\
& \qquad U_{h,ijkl}^{(s)}\left(\lambda\right) = \frac{\left(\lambda' U_{h,ij}\right)^s \left(\lambda' U_{h,kl}\right)^s + \left(\lambda' U_{h,ik}\right)^s \left(\lambda' U_{h,jl}\right)^s + \left(\lambda' U_{h,il}\right)^s \left(\lambda' U_{h,jk}\right)^s}{3}.
\end{aligned}$$

3.1.1 Term: $\hat{\Sigma}_n(h)$

Lemma 3.1.1:

$$\frac{1}{n}\lambda'\hat{\Sigma}_n(h)\lambda = 2\binom{n}{2}^{-1}\tilde{T}^{(2)}_{1,n}(\lambda;h) + \frac{4}{n}\frac{n-2}{n-1}\tilde{T}^{(1)}_{2,n}(\lambda;h) - \frac{4}{n}\left(\tilde{T}^{(1)}_{1,n}(\lambda;h)\right)^2.$$

Proof of Lemma 3.1.1:

$$\lambda'\hat{\Sigma}_n\lambda = \frac{4}{n}\sum_{i=1}^{n}\left(\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}\lambda' U_{h,ij} - \lambda'\hat{\theta}_n(h)\right)^2 = \frac{4}{n}\sum_{i=1}^{n}\left(\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}\lambda' U_{h,ij}\right)^2 - 4\left(\lambda'\hat{\theta}_n(h)\right)^2,$$

and

$$\begin{aligned}\frac{4}{n(n-1)^2}\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\lambda' U_{h,ij}\right)^2 &= \frac{4}{n(n-1)^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)\\ &= \frac{4}{n(n-1)^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2\\ &\quad+\frac{4}{n(n-1)^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right),\end{aligned}$$

with

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)\\ &= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\frac{\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)+\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)+\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)}{3}\\ &= 6\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\frac{\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)+\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)+\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)}{3}.\end{aligned}$$

This completes the proof. ■

3.1.2 Term: $\hat{\Delta}_n(h)$

Lemma 3.1.2:

$$\begin{aligned}\lambda'\hat{\Delta}_n(h)\lambda &= \left(1-\frac{2}{n-1}+\frac{2}{(n-1)^2}\right)h^{d+2}\tilde{T}^{(2)}_{1,n}(\lambda;h) - \left(\frac{2(n-2)}{n-1}-\frac{6(n-2)}{(n-1)^2}\right)h^{d+2}\tilde{T}^{(1)}_{2,n}(\lambda;h)\\ &\quad+\frac{2(n-2)(n-3)}{(n-1)^2}h^{d+2}\tilde{T}^{(1)}_{3,n}(\lambda;h) - h^{d+2}\left(\tilde{T}^{(1)}_{1,n}(\lambda;h)\right)^2,\end{aligned}$$

$$\lambda'\hat{\Delta}_{2,n}\lambda = h^{d+2}\tilde{T}^{(2)}_{1,n}(\lambda;h) - h^{d+2}\left(\tilde{T}^{(1)}_{1,n}(\lambda;h)\right)^2,$$

$$\lambda'\hat{\Delta}_{3,n}\lambda = h^{d+2}\tilde{T}^{(2)}_{1,n}(\lambda;h).$$

Proof of Lemma 3.1.2: Note that

$$\lambda'\hat{\Delta}_n(h)\lambda = \frac{h^{d+2}}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda'U_{h,ij} - \frac{1}{n-1}\sum_{k=1,k\neq i}^{n}\lambda'U_{h,ik} - \frac{1}{n-1}\sum_{l=1,l\neq j}^{n}\lambda'U_{h,jl} + \lambda'\hat{\theta}_n(h)\right)^2,$$

where

$$\begin{aligned}
S_1 &:= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda'U_{h,ij} - \frac{1}{n-1}\sum_{k=1,k\neq i}^{n}\lambda'U_{h,ik} - \frac{1}{n-1}\sum_{l=1,l\neq j}^{n}\lambda'U_{h,jl} + \lambda'\hat{\theta}_n(h)\right)^2 \\
&= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda'U_{h,ij}\right)^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\frac{1}{n-1}\sum_{k=1,k\neq i}^{n}\lambda'U_{h,ik}\right)^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\frac{1}{n-1}\sum_{l=1,l\neq j}^{n}\lambda'U_{h,jl}\right)^2 \\
&+ n(n-1)\left(\lambda'\hat{\theta}_n(h)\right)^2 \\
&- \frac{2}{n-1}\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\lambda'U_{h,ij}\right)^2 \\
&- \frac{2}{n-1}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1,l\neq j}^{n}\left(\lambda'U_{h,ij}\right)\left(\lambda'U_{h,jl}\right) + 2\left(\lambda'\hat{\theta}_n(h)\right)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\lambda'U_{h,ij} \\
&+ \frac{2}{(n-1)^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n}\sum_{l=1,l\neq j}^{n}\left(\lambda'U_{h,ik}\right)\left(\lambda'U_{h,jl}\right) - \frac{2}{n-1}\left(\lambda'\hat{\theta}_n(h)\right)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n}\lambda'U_{h,ik} \\
&- \frac{2}{n-1}\left(\lambda'\hat{\theta}_n(h)\right)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1,l\neq j}^{n}\lambda'U_{h,jl}.
\end{aligned}$$

Next, since

$$\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\frac{1}{n-1}\sum_{k=1,k\neq i}^{n}\lambda'U_{h,ik}\right)^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\lambda'U_{h,ij}\right)^2,$$

$$2\left(\lambda'\hat{\theta}_n\right)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\lambda'U_{h,ij} = 2n(n-1)\left(\lambda'\hat{\theta}_n\right)^2,$$

$$\frac{2}{n-1}\left(\lambda'\hat{\theta}_n\right)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n}\lambda'U_{h,ik} = 2n(n-1)\left(\lambda'\hat{\theta}_n\right)^2,$$

$$\begin{aligned}
\frac{2}{n-1}\left(\lambda'\hat{\theta}_n\right)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1,l\neq j}^{n}\lambda'U_{h,jl} &= \frac{2}{n-1}\left(\lambda'\hat{\theta}_n\right)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\lambda'U_{h,ji} \\
&+\frac{2}{n-1}\left(\lambda'\hat{\theta}_n\right)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1,l\neq i,l\neq j}^{n}\lambda'U_{h,jl} \\
&= 2n\left(\lambda'\hat{\theta}_n\right) + 2n(n-2)\left(\lambda'\hat{\theta}_n\right) = 2n(n-1)\left(\lambda'\hat{\theta}_n\right)
\end{aligned}$$

we have

$$
\begin{aligned}
S_1 = & \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \left(\lambda' U_{h,ij}\right)^{s} - n\left(n-1\right)\left(\lambda'\hat{\theta}_n\right)^2 \\
& - \frac{1}{n-1}\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\lambda' U_{h,ij}\right)^2 - \frac{2}{n-1}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right) \\
& + \frac{1}{\left(n-1\right)^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\sum_{k=1,k\neq j}^{n}\lambda' U_{h,jk}\right)^2 + \frac{2}{\left(n-1\right)^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n}\sum_{l=1,l\neq j}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right).
\end{aligned}
$$

Next, note that

$$
\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\lambda' U_{h,ij}\right)^2 = \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right),
$$

and

$$
\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right) = \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right),
$$

and therefore

$$
\begin{aligned}
S_1 = & \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \left(\lambda' U_{h,ij}\right)^{2} - n\left(n-1\right)\left(\lambda'\hat{\theta}_n\right)^2 \\
& - \frac{3}{n-1}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2 - \frac{3}{n-1}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right) \\
& + \frac{1}{\left(n-1\right)^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\sum_{k=1,k\neq j}^{n}\lambda' U_{h,jk}\right)^2 + \frac{2}{\left(n-1\right)^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n}\sum_{l=1,l\neq j}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right).
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\sum_{k=1,k\neq j}^{n}\lambda' U_{h,jk}\right)^2 &= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq j}^{n}\left(\lambda' U_{h,jk}\right)^2+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq j}^{n}\sum_{l=1,l\neq j,l\neq k}^{n}\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,jl}\right)\\
&= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2+(n-2)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2\\
&\quad+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1,l\neq j,l\neq i}^{n}\left(\lambda' U_{h,ji}\right)\left(\lambda' U_{h,jl}\right)\\
&\quad+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,ji}\right)\\
&\quad+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq j,l\neq k}^{n}\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,jl}\right)\\
&= (n-1)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2+(n-1)\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq j}^{n}\left(\lambda' U_{ij}\right)\left(\lambda' U_{h,ik}\right),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
S_1 &= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2-n(n-1)\left(\lambda'\hat{\theta}_n\right)^2\\
&\quad-\frac{2}{n-1}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2-\frac{2}{n-1}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)\\
&\quad+\frac{2}{(n-1)^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n}\sum_{l=1,l\neq j}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right).
\end{aligned}
$$

Finally, note that

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i}^{n}\sum_{l=1,l\neq j}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right)\\
&= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1,l\neq i,l\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jl}\right)\\
&\quad+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,ji}\right)+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)\\
&\quad+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq j,l\neq k}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right)\\
&= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2+3\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1,l\neq i,l\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jl}\right)\\
&\quad+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq j,l\neq k}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right),
\end{aligned}
$$

and therefore

$$\begin{aligned}
S_1 = & \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \left(\lambda' U_{h,ij}\right)^2 - n\left(n-1\right)\left(\lambda'\hat{\theta}_n\right)^2 \\
& - \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \left(\lambda' U_{h,ij}\right)^2 - \frac{2}{n-1} \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{k=1,k\neq i,k\neq j}^{n} \left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right) \\
& + \frac{2}{\left(n-1\right)^2} \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \left(\lambda' U_{h,ij}\right)^2 + \frac{6}{\left(n-1\right)^2} \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{k=1,k\neq i,k\neq j}^{n} \left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right) \\
& + \frac{2}{\left(n-1\right)^2} \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} \sum_{k=1,k\neq i,k\neq j}^{n} \sum_{l=1,l\neq i,l\neq j,l\neq k}^{n} \left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\lambda'\hat{\Delta}_n\left(h\right)\lambda = & \frac{h^{d+2}}{n\left(n-1\right)} S_1 \\
= & \frac{2h^{d+2}}{n\left(n-1\right)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\lambda' U_{h,ij}\right)^2 \\
& - h^{d+2}\left(\lambda'\hat{\theta}_n\right)^2 \\
& - \frac{4h^{d+2}}{n\left(n-1\right)^2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\lambda' U_{h,ij}\right)^2 \\
& - \frac{12h^{d+2}}{n\left(n-1\right)^2} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \frac{\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right) + \left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right) + \left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)}{3} \\
& + \frac{4h^{d+2}}{n\left(n-1\right)^3} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\lambda' U_{h,ij}\right)^2 \\
& + \frac{36h^{d+2}}{n\left(n-1\right)^3} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \frac{\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right) + \left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right) + \left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)}{3} \\
& + \frac{48h^{d+2}}{n\left(n-1\right)^3} \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} \frac{\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,kl}\right) + \left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right) + \left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,jk}\right)}{3}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& \lambda'\hat{\Delta}_n(h)\lambda \\
= \; & h^{d+2}\left(\frac{2}{n(n-1)}-\frac{4}{n(n-1)^2}+\frac{4}{n(n-1)^3}\right)\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\lambda U_{h,ij})^2 \\
& -h^{d+2}\left(\frac{12}{n(n-1)^2}-\frac{36}{n(n-1)^3}\right)\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\frac{(\lambda'U_{h,ij})(\lambda'U_{h,jk})+(\lambda'U_{h,ij})(\lambda'U_{h,ik})+(\lambda'U_{h,ik})(\lambda'U_{h,jk})}{3} \\
& +\frac{48h^{d+2}}{n(n-1)^3}\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n}\frac{(\lambda'U_{h,ij})(\lambda'U_{h,kl})+(\lambda'U_{h,ik})(\lambda'U_{h,jl})+(\lambda'U_{h,il})(\lambda'U_{h,jk})}{3} \\
& -h^{d+2}\left(\lambda'\hat{\theta}_n\right)^2,
\end{aligned}
$$

and this completes the proof for $\hat{\Delta}_n(h)$. The results for $\hat{\Delta}_{2,n}$ and $\hat{\Delta}_{3,n}$ follow by definition of $\tilde{T}_{1,n}^{(s)}(\lambda;h)$. ■

We need to analyze the stochastic properties of the n-varying U-statistics introduced earlier.

3.1.3 Term: $\tilde{T}_{1,n}^{(s)}(\lambda;h)$

Lemma 3.1.3:

$$
\tilde{T}_{1,n}^{(s)}(\lambda;h)=\mathbb{E}\left[(\lambda'U_{h,ij})^s\right]+O_p\left(n^{-1/2}h^{-(s-1)d-s\mathbf{1}(s>1)}+n^{-1}h^{-(2s-1)d/2-s}\right)
$$

for $s\in\{1,2\}$.

Proof of Lemma 3.1.3: Using the Hoeffding decomposition,

$$
\tilde{T}_{1,n}^{(s)}(\lambda;h)=\binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\lambda'U_{h,ij})^s=\mathbb{E}\left[(\lambda'U_{h,ij})^s\right]+\tilde{T}_{11,n}^{(s)}(\lambda;h)+\tilde{T}_{12,n}^{(s)}(\lambda;h),
$$

where

$$
\tilde{T}_{11,n}^{(s)}(\lambda;h)=n^{-1}\sum_{i=1}^{n}2\left(\mathbb{E}\left[(\lambda'U_{h,ij})^s|z_i\right]-\mathbb{E}\left[(\lambda'U_{h,ij})^s\right]\right),
$$

$$
\tilde{T}_{12,n}^{(s)}(\lambda;h)=\binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\lambda'U_{h,ij})^s-\mathbb{E}\left[(\lambda'U_{h,ij})^s|z_i\right]-\mathbb{E}\left[(\lambda'U_{h,ij})^s|z_j\right]+\mathbb{E}\left[(\lambda'U_{h,ij})^s\right]\right).
$$

Now,

$$
\begin{aligned}
\mathbb{E}\left[\left(\tilde{T}_{11,n}^{(s)}(\lambda;h)\right)^2\right] &= 4n^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[(\lambda'U_{h,ij})^s|z_i\right]-\mathbb{E}\left[(\lambda'U_{h,ij})^s\right]\right)^2\right]\le Cn^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[(\lambda'U_{h,ij})^s|z_i\right]\right)^2\right] \\
&= O\left(n^{-1}h^{-2(s-1)d-2s\mathbf{1}(s>1)}\right),
\end{aligned}
$$

by Lemma 2.2 with $\sigma = s$. Note that when $s = 2$, we require $\mathbb{E}|y_i|^4 < \infty$. Next,

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{T}^{(s)}_{12,n}(\lambda;h)\right)^2\right] &= \binom{n}{2}^{-1}\mathbb{E}\left[\left(\left(\lambda' U_{h,ij}\right)^s - \mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^s|z_i\right] - \mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^s|z_j\right] + \mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^s\right]\right)^2\right] \\ &\leq Cn^{-2}\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^{2s}\right] \\ &= O\left(n^{-2}h^{-(2s-1)d-2s}\right),\end{aligned}$$

by Lemma 2.1 with $\sigma = 2s$. Note that when $s = 2$, we require $\mathbb{E}|y_i|^4 < \infty$. ■

3.1.4 Term: $\tilde{T}^{(1)}_{2,n}(\lambda;h)$

Lemma 3.1.4:

$$\tilde{T}^{(1)}_{2,n}(\lambda;h) = \mathbb{E}\left[\left(\mathbb{E}\left[\lambda' U_{h,ij}|z_i\right]\right)^2\right] + O_p\left(n^{-1/2} + n^{-1}h^{-d/2-2} + n^{-3/2}h^{-d-2}\right).$$

Proof of Lemma 3.1.4: Recall that

$$\tilde{T}^{(1)}_{2,n}(\lambda;h) = \binom{n}{3}^{-1}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}U^{(1)}_{ijk}(\lambda),$$

$$U^{(1)}_{ijk}(\lambda) = \frac{\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right) + \left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right) + \left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)}{3}.$$

We drop the superscript to save notation. Using the Hoeffding decomposition,

$$\tilde{T}_{2,n}(\lambda;h) = \mathbb{E}\left[U_{h,ijk}(\lambda)\right] + \tilde{T}_{21,n}(\lambda;h) + \tilde{T}_{22,n}(\lambda;h) + \tilde{T}_{23,n}(\lambda;h),$$

where

$$\mathbb{E}\left[U_{h,ijk}(\lambda)\right] = \mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)\right] = \mathbb{E}\left[\left(\mathbb{E}\left[\lambda' U_{h,ij}|z_i\right]\right)^2\right],$$

$$\tilde{T}_{21,n}(\lambda;h) = n^{-1}\sum_{i=1}^{n}3\left(\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i\right] - \mathbb{E}\left[U_{h,ijk}(\lambda)\right]\right),$$

$$\tilde{T}_{22,n}(\lambda;h) = \binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}3\left(\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i,z_j\right] - \mathbb{E}\left[U_{h,ijk}(\lambda)|z_i\right] - \mathbb{E}\left[U_{h,ijk}(\lambda)|z_j\right] + \mathbb{E}\left[U_{h,ijk}(\lambda)\right]\right),$$

$$\begin{aligned}&\tilde{T}_{23,n}(\lambda;h) \\ = &\binom{n}{3}^{-1}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}(U_{h,ijk}(\lambda) + \mathbb{E}\left[U_{h,ijk}(\lambda)|z_i\right] + \mathbb{E}\left[U_{h,ijk}(\lambda)|z_j\right] + \mathbb{E}\left[U_{h,ijk}(\lambda)|z_k\right] \\ &-\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i,z_j\right] - \mathbb{E}\left[U_{h,ijk}(\lambda)|z_i,z_k\right] - \mathbb{E}\left[U_{h,ijk}(\lambda)|z_j,z_k\right] - \mathbb{E}\left[U_{h,ijk}(\lambda)\right]).\end{aligned}$$

Now,

$$\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i\right] = \frac{1}{3}\left(\mathbb{E}\left[\lambda' U_{h,ij}|z_i\right]\right)^2 + \frac{2}{3}\mathbb{E}\left[\lambda' U_{h,ij}\mathbb{E}\left[\lambda' U_{h,jk}|z_j\right]|z_i\right]$$

Thus,

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{T}_{21,n}(\lambda;h)\right)^2\right] &= 9n^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i\right]-\mathbb{E}\left[U_{h,ijk}(\lambda)\right]\right)^2\right] \leq Cn^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i\right]\right)^2\right] \\ &\leq Cn^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[\lambda' U_{h,ij}|z_i\right]\right)^4\right]+Cn^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[\lambda' U_{h,ij}\mathbb{E}\left[\lambda' U_{h,jk}|z_j\right]|z_i\right]\right)^2\right] \\ &= O\left(n^{-1}\right),\end{aligned}$$

where the first term follows by Lemma 2.2 (note we require $\mathbb{E}|y_i|^4<\infty$) and the second term follows by Lemma 2.3.

Similarly,

$$\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i,z_j\right]=\frac{1}{3}\left(\lambda' U_{h,ij}\right)\mathbb{E}\left[\lambda' U_{h,ik}|z_i\right]+\frac{1}{3}\left(\lambda' U_{h,ij}\right)\mathbb{E}\left[\lambda' U_{h,jk}|z_j\right]+\frac{1}{3}\mathbb{E}\left[\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)|z_i,z_j\right],$$

and hence

$$\begin{aligned}&\mathbb{E}\left[\left(\tilde{T}_{22,n}(\lambda;h)\right)^2\right] \\ =\ & 9\binom{n}{2}^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i,z_j\right]-\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i\right]-\mathbb{E}\left[U_{h,ijk}(\lambda)|z_j\right]+\mathbb{E}\left[U_{h,ijk}(\lambda)\right]\right)^2\right] \\ \leq\ & Cn^{-2}\mathbb{E}\left[\left(\mathbb{E}\left[U_{h,ijk}(\lambda)|z_i,z_j\right]\right)^2\right] \\ \leq\ & Cn^{-2}\mathbb{E}\left[\left(\left(\lambda' U_{h,ij}\right)\mathbb{E}\left[\lambda' U_{h,ik}|z_i\right]\right)^2\right]+Cn^{-2}\mathbb{E}\left[\left(\mathbb{E}\left[\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)|z_i,z_j\right]\right)^2\right] \\ =\ & O\left(n^{-2}h^{-d-2}+n^{-2}h^{-d-4}\right).\end{aligned}$$

The first term follows by repeated use of Lemma 2.2 and the second term follows by Lemma 2.4.

Finally,

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{T}_{23,n}(\lambda;h)\right)^2\right] &\leq C\binom{n}{3}^{-1}\mathbb{E}\left[\left(U_{h,ijk}(\lambda)\right)^2\right]\leq Cn^{-3}\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,ik}\right)^2\right] \\ &= Cn^{-3}\mathbb{E}\left[\left(\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^2|z_i\right]\right)^2\right]=O\left(n^{-3}h^{-2d-4}\right),\end{aligned}$$

follows by Lemma 2.2. ■

3.1.5 Term: $\tilde{T}^{(1)}_{3,n}(\lambda;h)$

Lemma 3.1.5:

$$\tilde{T}^{(1)}_{3,n}(\lambda;h)=\left(\mathbb{E}\left[\lambda' U_{h,ij}\right]\right)^2+O_p\left(n^{-1/2}+n^{-1}h^{-(d+2)/2}+n^{-2}h^{-(d+2)}\right).$$

Proof of Lemma 3.1.5: Recall that

$$\begin{aligned}\tilde{T}^{(1)}_{3,n}(\lambda;h) &= \binom{n}{4}^{-1}\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n}U^{(1)}_{h,ijkl}(\lambda), \\ & U^{(1)}_{h,ijkl}(\lambda)=\frac{\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,kl}\right)+\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right)+\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,jk}\right)}{3}.\end{aligned}$$

We drop the superscript to save notation. Using a Hoeffding decomposition,

$$\tilde{T}_{3,n}(\lambda;h) = \mathbb{E}[U_{h,ijkl}(\lambda)] + \tilde{T}_{31,n}(\lambda;h) + \tilde{T}_{32,n}(\lambda;h) + \tilde{T}_{33,n}(\lambda;h) + \tilde{T}_{34,n}(\lambda;h),$$

where

$$\tilde{T}_{31,n}(\lambda;h) = n^{-1}\sum_{i=1}^{n} 4\left(\mathbb{E}[U_{h,ijkl}(\lambda)|z_i] - \mathbb{E}[U_{h,ijkl}(\lambda)]\right),$$

$$\tilde{T}_{32,n}(\lambda;h) = \binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 6\left(\mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_j] - \mathbb{E}[U_{h,ijkl}(\lambda)|z_i] - \mathbb{E}[U_{h,ijkl}(\lambda)|z_j] + \mathbb{E}[U_{h,ijkl}(\lambda)]\right),$$

$$\begin{aligned}
&\tilde{T}_{33,n}(\lambda;h)\\
= &\binom{n}{3}^{-1}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} 4(\mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_j,z_k] + \mathbb{E}[U_{h,ijkl}(\lambda)|z_i] + \mathbb{E}[U_{h,ijkl}(\lambda)|z_j] + \mathbb{E}[U_{h,ijkl}(\lambda)|z_k]\\
&\qquad -\mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_j] - \mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_k] - \mathbb{E}[U_{h,ijkl}(\lambda)|z_k,z_j] - \mathbb{E}[U_{h,ijkl}(\lambda)]),
\end{aligned}$$

$$\begin{aligned}
\tilde{T}_{34,n}(\lambda;h) = &\binom{n}{4}^{-1}\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n}(U_{h,ijkl}(\lambda) + \mathbb{E}[U_{h,ijkl}(\lambda)]\\
&-\mathbb{E}[U_{h,ijkl}(\lambda)|z_i] - \mathbb{E}[U_{h,ijkl}(\lambda)|z_j] - \mathbb{E}[U_{h,ijkl}(\lambda)|z_k] - \mathbb{E}[U_{h,ijkl}(\lambda)|z_l]\\
&+\mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_j] + \mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_k] + \mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_l]\\
&+\mathbb{E}[U_{h,ijkl}(\lambda)|z_j,z_k] + \mathbb{E}[U_{h,ijkl}(\lambda)|z_j,z_l] + \mathbb{E}[U_{h,ijkl}(\lambda)|z_k,z_l]\\
&-\mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_j,z_k] - \mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_j,z_l]\\
&-\mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_k,z_l] - \mathbb{E}[U_{h,ijkl}(\lambda)|z_j,z_k,z_l]).
\end{aligned}$$

Note that

$$\mathbb{E}\left[\lambda' U_{h,ijkl}(\lambda)\right] = \left(\mathbb{E}\left[\lambda' U_{h,ij}\right]\right)^2, \qquad \mathbb{E}[U_{h,ijkl}(\lambda)|z_i] = \mathbb{E}\left[\lambda' U_{h,ik}|z_i\right]\mathbb{E}\left[\lambda' U_{h,jl}\right],$$

and hence

$$\mathbb{E}\left[\left(\tilde{T}_{31,n}(\lambda;h)\right)^2\right] \le Cn^{-1}\mathbb{E}\left[\left(\mathbb{E}[U_{h,ijkl}(\lambda)|z_i]\right)^2\right] = Cn^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[\lambda' U_{h,ik}|z_i\right]\right)^2\right]\left(\mathbb{E}\left[\lambda' U_{h,jl}\right]\right)^2 = O\left(n^{-1}\right),$$

follows by Lemma 2.1 and Lemma 2.2.

Next, note that

$$\begin{aligned}
&\mathbb{E}[U_{h,ijkl}(\lambda)|z_i,z_j]\\
= &\frac{1}{3}\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,kl}\right)|z_i,z_j\right] + \frac{1}{3}\mathbb{E}\left[\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right)|z_i,z_j\right] + \frac{1}{3}\mathbb{E}\left[\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,jk}\right)|z_i,z_j\right]\\
= &\frac{1}{3}\left(\lambda' U_{h,ij}\right)\mathbb{E}\left[\left(\lambda' U_{h,kl}\right)\right] + \frac{2}{3}\mathbb{E}\left[\lambda' U_{h,ik}|z_i\right]\mathbb{E}\left[\lambda' U_{h,jl}|z_j\right],
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbb{E}\left[\left(\tilde{T}_{32,n}(\lambda;h)\right)^2\right] &\leq C\binom{n}{2}^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[U_{h,ijkl}(\lambda)|z_i,z_j\right]\right)^2\right] \\
&\leq Cn^{-2}\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^2\right]\left(\mathbb{E}\left[\lambda' U_{h,kl}\right]\right)^2 + Cn^{-2}\left(\mathbb{E}\left[\left(\mathbb{E}\left[\lambda' U_{h,ik}|z_i\right]\right)^2\right]\right)^2 \\
&= O\left(n^{-2}h^{-d-2}\right).
\end{aligned}$$

The first term follows by Lemma 2.1 and the second term follows by Lemma 2.2.

Next, note that

$$\begin{aligned}
&\mathbb{E}\left[U_{h,ijkl}(\lambda)|z_i,z_j,z_k\right] \\
=\ &\frac{1}{3}\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,kl}\right)|z_i,z_j,z_k\right] + \frac{1}{3}\mathbb{E}\left[\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jl}\right)|z_i,z_j,z_k\right] + \frac{1}{3}\mathbb{E}\left[\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,jk}\right)|z_i,z_j,z_k\right] \\
=\ &\frac{1}{3}\left(\lambda' U_{h,ij}\right)\mathbb{E}\left[\lambda' U_{h,kl}|z_k\right] + \frac{1}{3}\left(\lambda' U_{h,ik}\right)\mathbb{E}\left[\lambda' U_{h,jl}|z_j\right] + \frac{1}{3}\left(\lambda' U_{h,jk}\right)\mathbb{E}\left[\lambda' U_{h,il}|z_i\right],
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbb{E}\left[\left(\tilde{T}_{33,n}(\lambda;h)\right)^2\right] &\leq C\binom{n}{3}^{-1}\mathbb{E}\left[\left(\mathbb{E}\left[U_{h,ijkl}(\lambda)|z_i,z_j,z_k\right]\right)^2\right] \leq Cn^{-3}\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^2\left(\mathbb{E}\left[\lambda' U_{h,kl}|z_k\right]\right)^2\right] \\
&= Cn^{-3}\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^2\right]\mathbb{E}\left[\left(\mathbb{E}\left[\lambda' U_{h,kl}|z_k\right]\right)^2\right] \\
&= O\left(n^{-3}h^{-d-2}\right),
\end{aligned}$$

which follows by Lemma 2.1 and Lemma 2.2.

Finally, note that

$$U_{h,ijkl}(\lambda)^2 \leq C\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,kl}\right)^2 + C\left(\lambda' U_{h,ik}\right)^2\left(\lambda' U_{h,jl}\right)^2 + C\left(\lambda' U_{h,il}\right)^2\left(\lambda' U_{h,jk}\right)^2,$$

and hence

$$\mathbb{E}\left[\left(\tilde{T}_{34,n}(\lambda;h)\right)^2\right] \leq C\binom{n}{4}^{-1}\mathbb{E}\left[U_{h,ijkl}(\lambda)^2\right] \leq Cn^{-4}\left(\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^2\right]\right)^2 = O\left(n^{-4}h^{-2d-4}\right),$$

by Lemma 2.1. This completes the proof. ■

3.1.6 Rates of Convergence in Probability of $\hat{\Sigma}_n$ and $\hat{\Delta}_n$

In Cattaneo, Crump and Jansson (2011) it was shown that,

$$\frac{1}{n}\lambda'\hat{\Sigma}_n(h)\lambda = \frac{1}{n}\left[\lambda'\Sigma\lambda + o_p(1)\right] + 2\binom{n}{2}^{-1}h_n^{-(d+2)}\left[\lambda'\Delta\lambda + o_p(1)\right].$$

This result now follows immediately from the properties of the above U-statistics. This decomposition will greatly simplify our calculations of the bootstrap analogs. By Lemma 3.1.1,

$$\frac{1}{n}\lambda'\hat{\Sigma}_n(h)\lambda = \frac{4}{n}\frac{n-2}{n-1}\tilde{T}_{2,n}^{(1)}(\lambda;h) - \frac{4}{n}\left(\tilde{T}_{1,n}^{(1)}(\lambda;h)\right)^2 + 2\binom{n}{2}^{-1}h^{-(d+2)}h^{d+2}\tilde{T}_{1,n}^{(2)}(\lambda;h),$$

where

$$\tilde{T}^{(1)}_{1,n}(\lambda;h) = \mathbb{E}\left[\lambda' U_{h,ij}\right] + O_p\left(n^{-1/2} + n^{-1}h^{-d/2-1}\right),$$

$$\tilde{T}^{(2)}_{1,n}(\lambda;h) = \mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^2\right] + O_p\left(n^{-1/2}h^{-d-2} + n^{-1}h^{-3d/2-2}\right),$$

$$\tilde{T}^{(1)}_{2,n}(\lambda;h) = \mathbb{E}\left[\left(\mathbb{E}\left[\lambda' U_{h,ij}|z_i\right]\right)^2\right] + O_p\left(n^{-1/2} + n^{-1}h^{-d/2-2} + n^{-3/2}h^{-d-2}\right),$$

by Lemmas 3.1.3 and 3.1.4, and therefore,

$$\begin{aligned}
&\frac{1}{n}\lambda'\hat{\Sigma}_n(h)\lambda\Big|_{h=h_n} \\
=\ & \frac{1}{n}\lambda'\hat{\Sigma}_n\lambda \\
=\ & \frac{4}{n}\frac{n-2}{n-1}\tilde{T}^{(1)}_{2,n}(\lambda;h_n) - \frac{4}{n}\left(\tilde{T}^{(1)}_{1,n}(\lambda;h_n)\right)^2 + 2\binom{n}{2}^{-1} h_n^{-(d+2)} h_n^{d+2}\tilde{T}^{(2)}_{1,n}(\lambda;h_n) \\
=\ & \frac{4}{n}\left[\mathbb{E}\left[\left(\mathbb{E}\left[\lambda' U_{h,ij}|z_i\right]\right)^2\right] + O_p\left(n^{-1/2} + n^{-1}h_n^{-d/2-2} + n^{-3/2}h_n^{-d-2}\right)\right] \\
&\quad -\frac{4}{n}\left[\mathbb{E}\left[\lambda' U_{h,ij}\right] + O_p\left(n^{-1/2} + n^{-1}h_n^{-d/2-1}\right)\right]^2 \\
&\quad +2\binom{n}{2}^{-1} h_n^{-(d+2)}\left[h_n^{d+2}\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^2\right] + O_p\left(n^{-1/2} + n^{-1}h_n^{-d/2}\right)\right] \\
=\ & \frac{1}{n}\left[\lambda'\Sigma\lambda + o_p(1)\right] + 2\binom{n}{2}^{-1} h_n^{-(d+2)}\left[\lambda'\Delta\lambda + o_p(1)\right], \qquad \text{if } n^2h_n^d \to \infty.
\end{aligned}$$

Similarly, by Lemma 3.1.5,

$$\tilde{T}^{(1)}_{3,n}(\lambda;h) = \left(\mathbb{E}\left[\lambda' U_{h,ij}\right]\right)^2 + O_p\left(n^{-1/2} + n^{-1}h^{-(d+2)/2} + n^{-2}h^{-(d+2)}\right)$$

and so by Lemma 3.1.2, it follows that

$$\begin{aligned}
&\lambda'\hat{\Delta}_n(h)\lambda\Big|_{h=h_n} \\
=\ & \lambda'\hat{\Delta}_n\lambda \\
=\ & \left(1 - \frac{2}{(n-1)} + \frac{2}{(n-1)^2}\right) h_n^{d+2}\tilde{T}^{(2)}_{1,n}(\lambda;h_n) - \left(\frac{2(n-2)}{n-1} - \frac{6(n-2)}{(n-1)^2}\right) h^{d+2}\tilde{T}^{(1)}_{2,n}(\lambda;h_n) \\
&+\frac{2(n-2)(n-3)}{(n-1)^2} h^{d+2}\tilde{T}^{(1)}_{3,n}(\lambda;h_n) - h^{d+2}\left(\tilde{T}^{(1)}_{1,n}(\lambda;h_n)\right)^2 \\
=\ & \lambda'\Delta\lambda + o_p(1), \qquad \text{if } n^2h_n^d \to \infty.
\end{aligned}$$

$$\lambda'\hat{\Delta}_{2,n}(h)\lambda\Big|_{h=h_n} = \lambda'\hat{\Delta}_{2,n}\lambda = h_n^{d+2}\tilde{T}^{(2)}_{1,n}(\lambda;h_n) - h^{d+2}\left(\tilde{T}^{(1)}_{1,n}(\lambda;h_n)\right)^2 = \lambda'\Delta\lambda + o_p(1), \qquad \text{if } n^2h_n^d \to \infty.$$

$$\lambda'\hat{\Delta}_{3,n}(h)\lambda\Big|_{h=h_n} = \lambda'\hat{\Delta}_{3,n}\lambda = h_n^{d+2}\tilde{T}^{(2)}_{1,n}(\lambda;h_n) = \Delta + o_p(1), \qquad \text{if } n^2h_n^d \to \infty.$$

3.1.7 Additional n-varying U-statistics

We will also need to define a number of other n-varying U-statistics. Define,

$$\tilde{T}_{4,n}(\lambda;h)=\binom{n}{4}^{-1}\sum_{1\leq i<j<k<l\leq n}U_{4,h,ijkl}(\lambda),$$

$$\begin{aligned}
&U_{4,h,ijkl}(\lambda)\\
=\;&\frac{(\lambda'U_{h,ij})^2(\lambda'U_{h,ik})(\lambda'U_{h,il})+(\lambda'U_{h,ik})^2(\lambda'U_{h,ij})(\lambda'U_{h,il})+(\lambda'U_{h,il})^2(\lambda'U_{h,ik})(\lambda'U_{h,ij})}{12}\\
&+\frac{(\lambda'U_{h,ji})^2(\lambda'U_{h,jk})(\lambda'U_{h,jl})+(\lambda'U_{h,jk})^2(\lambda'U_{h,ji})(\lambda'U_{h,jl})+(\lambda'U_{h,jl})^2(\lambda'U_{h,jk})(\lambda'U_{h,ji})}{12}\\
&+\frac{(\lambda'U_{h,kj})^2(\lambda'U_{h,ki})(\lambda'U_{h,kl})+(\lambda'U_{h,ki})^2(\lambda'U_{h,kj})(\lambda'U_{h,kl})+(\lambda'U_{h,kl})^2(\lambda'U_{h,ki})(\lambda'U_{h,kj})}{12}\\
&+\frac{(\lambda'U_{h,lj})^2(\lambda'U_{h,lk})(\lambda'U_{h,li})+(\lambda'U_{h,lk})^2(\lambda'U_{h,lj})(\lambda'U_{h,li})+(\lambda'U_{h,li})^2(\lambda'U_{h,lk})(\lambda'U_{h,lj})}{12},
\end{aligned}$$

and

$$\tilde{T}_{5,n}(\lambda;h)=\binom{n}{4}^{-1}\sum_{1\leq i<j<k<l\leq n}U_{5,h,ijkl}(\lambda),$$

$$\begin{aligned}
&U_{5,h,ijkl}(\lambda)\\
=\;&\frac{(\lambda'U_{h,ij})(\lambda'U_{h,ik})(\lambda'U_{h,il})(\lambda'U_{h,jk})+(\lambda'U_{h,il})(\lambda'U_{h,ik})(\lambda'U_{h,ij})(\lambda'U_{h,lk})+(\lambda'U_{h,ij})(\lambda'U_{h,il})(\lambda'U_{h,ik})(\lambda'U_{h,jl})}{12}\\
&+\frac{(\lambda'U_{h,ji})(\lambda'U_{h,jk})(\lambda'U_{h,jl})(\lambda'U_{h,ik})+(\lambda'U_{h,jl})(\lambda'U_{h,jk})(\lambda'U_{h,ji})(\lambda'U_{h,lk})+(\lambda'U_{h,ji})(\lambda'U_{h,jl})(\lambda'U_{h,jk})(\lambda'U_{h,il})}{12}\\
&+\frac{(\lambda'U_{h,kj})(\lambda'U_{h,ki})(\lambda'U_{h,kl})(\lambda'U_{h,ji})+(\lambda'U_{h,kl})(\lambda'U_{h,ki})(\lambda'U_{h,kj})(\lambda'U_{h,li})+(\lambda'U_{h,kj})(\lambda'U_{h,kl})(\lambda'U_{h,ki})(\lambda'U_{h,jl})}{12}\\
&+\frac{(\lambda'U_{h,lj})(\lambda'U_{h,lk})(\lambda'U_{h,li})(\lambda'U_{h,jk})+(\lambda'U_{h,li})(\lambda'U_{h,lk})(\lambda'U_{h,lj})(\lambda'U_{h,ik})+(\lambda'U_{h,lj})(\lambda'U_{h,li})(\lambda'U_{h,lk})(\lambda'U_{h,ji})}{12},
\end{aligned}$$

and

$$\tilde{T}_{6,n}(\lambda;h)=\binom{n}{4}^{-1}\sum_{1\leq i<j<k<l\leq n}U_{6,h,ijkl}(\lambda),$$

$$\begin{aligned}
U_{6,h,ijkl}(\lambda)\;=\;&\frac{(\lambda'U_{h,ij})(\lambda'U_{h,jk})(\lambda'U_{h,il})(\lambda'U_{h,kl})+(\lambda'U_{h,ij})(\lambda'U_{h,jl})(\lambda'U_{h,ik})(\lambda'U_{h,kl})}{3}\\
&+\frac{(\lambda'U_{h,ki})(\lambda'U_{h,il})(\lambda'U_{h,kj})(\lambda'U_{h,lj})}{3},
\end{aligned}$$

and

$$\tilde{T}_{7,n}(\lambda;h)=\binom{n}{5}^{-1}\sum_{1\leq i<j<k<l<r\leq n}U_{7,h,ijklr}(\lambda),$$

$$\begin{aligned}U_{7,h,ijklr}(\lambda) &= \frac{(\lambda' U_{h,ij})(\lambda' U_{h,ik})(\lambda' U_{h,il})(\lambda' U_{h,ir})}{5}+\frac{(\lambda' U_{h,ji})(\lambda' U_{h,jk})(\lambda' U_{h,jl})(\lambda' U_{h,jr})}{5}\\ &+\frac{(\lambda' U_{h,kj})(\lambda' U_{h,ki})(\lambda' U_{h,kl})(\lambda' U_{h,kr})}{5}\\ &+\frac{(\lambda' U_{h,lj})(\lambda' U_{h,lk})(\lambda' U_{h,li})(\lambda' U_{h,lr})}{5}+\frac{(\lambda' U_{h,rj})(\lambda' U_{h,rk})(\lambda' U_{h,rl})(\lambda' U_{h,ri})}{5},\end{aligned}$$

and

$$\tilde{T}_{8,n}(\lambda;h)=\binom{n}{5}^{-1}\sum_{1\leq i<j<k<l<r\leq n}U_{8,h,ijklr}(\lambda),$$

$U_{8,h,ijklr}(\lambda)$ is comprised of 60 terms of the form $(\lambda' U_{h,ij})(\lambda' U_{h,jk})(\lambda' U_{h,il})(\lambda' U_{h,lr})$. We omit the exact expression for brevity. Finally, define

$$\tilde{T}_{9,n}(\lambda;h)=\binom{n}{3}^{-1}\sum_{1\leq i<j<k\leq n}U_{9,h,ijk}(\lambda),$$

$$\begin{aligned}&U_{9,h,ijk}(\lambda)\\ =\ &\frac{(\lambda' U_{h,ij})(\lambda' U_{h,jk})^2(\lambda' U_{h,ik})+(\lambda' U_{h,ik})(\lambda' U_{h,ij})^2(\lambda' U_{h,jk})+(\lambda' U_{h,ij})(\lambda' U_{h,ik})^2(\lambda' U_{h,jk})}{3}.\end{aligned}$$

3.2 Moment Bounds

3.2.1 Term: $\mathbb{E}[\tilde{T}^{(s)}_{1,n}(\lambda;h)]$

Lemma 3.2.1:

$$\mathbb{E}\left[\tilde{T}^{(s)}_{1,n}(\lambda;h)\right]=O\left(h^{-(s-1)d-s\mathbf{1}(s>1)}\right).$$

Proof of Lemma 3.2.1: The result follows by

$$\left|\mathbb{E}\left[\tilde{T}^{(s)}_{1,n}(\lambda;h)\right]\right|=\left|\mathbb{E}\left[\binom{n}{2}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\lambda' U_{h,ij})^s\right]\right|=\left|\mathbb{E}\left[(\lambda' U_{h,ij})^s\right]\right|=O\left(h^{-(s-1)d-s\mathbf{1}(s>1)}\right),$$

by Lemma 2.1. Note when $s=4$ we require $\mathbb{E}|y_i|^4<\infty$.

3.2.2 Term: $\mathbb{E}[\tilde{T}^{(s)}_{2,n}(\lambda;h)]$

Lemma 3.2.2:

$$\mathbb{E}\left[\tilde{T}^{(s)}_{2,n}(\lambda;h)\right] = O\left(h^{-2(s-1)d-2s\mathbf{1}(s>1)}\right),$$

for $s \in \{1,2\}$.

Proof of Lemma 3.2.2: The result follows by,

$$\left|\mathbb{E}\left[\tilde{T}^{(s)}_{2,n}(\lambda;h)\right]\right| = \left|\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^s \left(\lambda' U_{h,ik}\right)^s\right]\right| = \left|\mathbb{E}\left[\left(\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^s \middle| z_i\right]\right)^2\right]\right| = O\left(h^{-2(s-1)d-2s\mathbf{1}(s>1)}\right),$$

by Lemma 2.2. Note when $s=2$ we require $\mathbb{E}|y_i|^4 < \infty$.

3.2.3 Term: $\mathbb{E}[\tilde{T}^{(1)}_{3,n}(\lambda;h)]$

Lemma 3.2.3:

$$\mathbb{E}\left[\tilde{T}^{(1)}_{3,n}(\lambda;h)\right] = O(1).$$

Proof of Lemma 3.2.3: The result follows by,

$$\left|\mathbb{E}\left[\tilde{T}^{(1)}_{3,n}(\lambda;h)\right]\right| = \left|\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,kl}\right)\right]\right| = \left|\mathbb{E}\left[\lambda' U_{h,ij}\right]\right|\left|\mathbb{E}\left[\lambda' U_{h,kl}\right]\right| = O(1),$$

by Lemma 2.1.

3.2.4 Term: $\mathbb{E}[\tilde{T}_{4,n}(\lambda;h)]$

Lemma 3.2.4:

$$\mathbb{E}\left[\tilde{T}_{4,n}(\lambda;h)\right] = O\left(h^{-d-2}\right).$$

Proof of Lemma 3.2.4: The result follows by,

$$\begin{aligned}\left|\mathbb{E}\left[\tilde{T}_{4,n}(\lambda;h)\right]\right| &= \left|\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,il}\right)\right]\right| = \left|\mathbb{E}\left[\mathbb{E}\left[\left|\lambda' U_{h,ij}\right|^2 \middle| z_i\right]\mathbb{E}\left[\lambda' U_{h,ik} \middle| z_i\right]\mathbb{E}\left[\lambda' U_{h,il} \middle| z_i\right]\right]\right| \\ &= O\left(h^{-d-2}\right),\end{aligned}$$

by Lemma 2.2. Note this result requires $\mathbb{E}|y_i|^4 < \infty$.

3.2.5 Term: $\mathbb{E}[\tilde{T}_{5,n}(\lambda;h)]$

Lemma 3.2.5:

$$\left|\mathbb{E}\left[\tilde{T}_{5,n}(\lambda;h)\right]\right| = O\left(h^{-d-3}\right).$$

Proof of Lemma 3.2.5: The result follows by,

$$\begin{aligned}
\left|\mathbb{E}\left[\tilde{T}_{5,n}(\lambda;h)\right]\right| &= \left|\mathbb{E}\left[(\lambda' U_{h,ij})(\lambda' U_{h,ik})(\lambda' U_{h,il})(\lambda' U_{h,jk})\right]\right| \\
&\leq \left(\mathbb{E}\left[\left(\mathbb{E}\left[(\lambda' U_{h,ij})(\lambda' U_{h,il})\middle| z_i, z_j\right]\right)^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left(\mathbb{E}\left[(\lambda' U_{h,ik})(\lambda' U_{h,jk})\middle| z_i, z_j\right]\right)^2\right]\right)^{1/2} \\
&= \left(\mathbb{E}\left[\left((\lambda' U_{h,ij})\,\mathbb{E}\left[(\lambda' U_{h,il})\middle| z_i\right]\right)^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left(\mathbb{E}\left[(\lambda' U_{h,ik})(\lambda' U_{h,jk})\middle| z_i, z_j\right]\right)^2\right]\right)^{1/2} \\
&= O\left(h^{-d/2-1}h^{-d/2-2}\right) \\
&= O\left(h^{-d-3}\right).
\end{aligned}$$

The first factor follows by repeated use of Lemma 2.2. The second factor follows by Lemma 2.4.

3.2.6 Term: $\mathbb{E}[\tilde{T}_{6,n}(\lambda;h)]$

Lemma 3.2.6:

$$\mathbb{E}\left[\tilde{T}_{6,n}(\lambda;h)\right] = O\left(h^{-d-4}\right).$$

Proof of Lemma 3.2.6: The result follows by,

$$\begin{aligned}
\left|\mathbb{E}\left[\tilde{T}_{6,n}(\lambda;h)\right]\right| &= \left|\mathbb{E}\left[(\lambda' U_{h,ij})(\lambda' U_{h,jk})(\lambda' U_{h,il})(\lambda' U_{h,kl})\right]\right| \\
&= \left|\mathbb{E}\left[\mathbb{E}\left[(\lambda' U_{h,ij})(\lambda' U_{h,jk})\middle| z_i, z_k\right]\mathbb{E}\left[(\lambda' U_{h,il})(\lambda' U_{h,kl})\middle| z_i, z_k\right]\right]\right| \\
&\leq \left(\mathbb{E}\left[\left(\mathbb{E}\left[(\lambda' U_{h,ij})(\lambda' U_{h,jk})\middle| z_i, z_k\right]\right)^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left(\mathbb{E}\left[(\lambda' U_{h,il})(\lambda' U_{h,kl})\middle| z_i, z_k\right]\right)^2\right]\right)^{1/2} \\
&= O\left(h^{-d-4}\right),
\end{aligned}$$

by Lemma 2.4.

3.2.7 Term: $\mathbb{E}[\tilde{T}_{7,n}(\lambda;h)]$

Lemma 3.2.7:

$$\mathbb{E}\left[\tilde{T}_{7,n}(\lambda;h)\right] = O(1).$$

Proof of Lemma 3.2.7: The result follows by,

$$\begin{aligned}
\left|\mathbb{E}\left[\tilde{T}_{7,n}(\lambda;h)\right]\right| &= \left|\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,ir}\right)\right]\right| \\
&= \left|\mathbb{E}\left[\mathbb{E}\left[\lambda' U_{h,ij}\middle| z_i\right]\mathbb{E}\left[\left|\lambda' U_{h,ik}\right|\middle| z_i\right]\mathbb{E}\left[\lambda' U_{h,il}\middle| z_i\right]\mathbb{E}\left[\lambda' U_{h,ir}\middle| z_i\right]\right]\right| \\
&= O(1),
\end{aligned}$$

by Lemma 2.2. Note this requires $\mathbb{E}\left|y_i\right|^4 < \infty$.

3.2.8 Term: $\mathbb{E}[\tilde{T}_{8,n}(\lambda;h)]$

Lemma 3.2.8:

$$\mathbb{E}\left[\tilde{T}_{8,n}(\lambda;h)\right] = O(1).$$

Proof of Lemma 3.2.8: The result follows by,

$$\begin{aligned}
\left|\mathbb{E}\left[\tilde{T}_{8,n}(\lambda;h)\right]\right| &= \left|\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,lr}\right)\right]\right| \\
&= \left|\mathbb{E}\left[\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)\middle| z_i\right]\mathbb{E}\left[\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,lr}\right)\middle| z_i\right]\right]\right| \\
&= \left|\mathbb{E}\left[\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\mathbb{E}\left[\left(\lambda' U_{h,jk}\right)\middle| z_j\right]\middle| z_i\right]\mathbb{E}\left[\left(\lambda' U_{h,il}\right)\mathbb{E}\left[\left(\lambda' U_{h,lr}\right)\middle| z_l\right]\middle| z_i\right]\right]\right| \\
&= O(1),
\end{aligned}$$

by Lemma 2.3.

3.2.9 Term: $\mathbb{E}[\tilde{T}_{9,n}(\lambda;h)]$

Lemma 3.2.9:

$$\mathbb{E}\left[\tilde{T}_{9,n}(\lambda;h)\right] = O\left(h^{-2d-4}\right).$$

Proof of Lemma 3.2.9: The result follows by,

$$\begin{aligned}
\mathbb{E}\left[\tilde{T}_{9,n}(\lambda;h)\right] &= \mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)^2\left(\lambda' U_{h,ik}\right)\right] \\
&\leq \left(\mathbb{E}\left[\left|\lambda' U_{h,jk}\right|^4\right]\right)^{1/2}\left(\mathbb{E}\left[\left(\mathbb{E}\left[\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)\middle| z_j, z_k\right]\right)^2\right]\right)^{1/2} \\
&= O\left(h^{-3/2d-2}\right)O\left(h^{-d/2-2}\right) \\
&= O\left(h^{-2d-4}\right),
\end{aligned}$$

by Lemma 2.1 and Lemma 2.4.

4 Bootstrap Statistics

4.1 Basic Properties

Since z_i^* is independent of z_j^*,

$$\mathbb{E}^*\left[U\left(z_i^*, z_j^*; h\right)\middle| z_i^*\right] = \sum_{j=1}^{n} U\left(z_i^*, z_j; h\right) n^{-1} = n^{-1}\sum_{j=1}^{n} U\left(z_i^*, z_j; h\right).$$

Also,

$$\theta^*(h) = \mathbb{E}^*\left[U\left(z_i^*, z_j^*; h\right)\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} U\left(z_i, z_j; h\right) n^{-2} = 2n^{-2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} U\left(z_i, z_j; h\right) = \frac{n-1}{n}\hat{\theta}_n(h),$$

where the second equality follows by $U(z_i, z_i; h) = 0$. Using these two results we have,

$$\begin{aligned}
L^*\left(z_i^*; h\right) &= 2\left[\mathbb{E}^*\left[U\left(z_i^*, z_j^*; h\right)\middle| z_i^*\right] - \theta^*(h)\right] \\
&= 2\left[n^{-1}\sum_{j=1}^{n} U\left(z_i^*, z_j; h\right) - \frac{n-1}{n}\hat{\theta}_n(h)\right] \\
&= 2\frac{n-1}{n}\left[(n-1)^{-1}\sum_{j=1}^{n} U\left(z_i^*, z_j; h\right) - \hat{\theta}_n(h)\right] \\
&= \left(\frac{n-1}{n}\right) 2\left[(n-1)^{-1}\sum_{j=1}^{n} U\left(z_i^*, z_j; h\right) - \hat{\theta}_n(h)\right] \\
&= \left(\frac{n-1}{n}\right)\hat{L}_n\left(z_i^*; h\right).
\end{aligned}$$

Clearly, $\mathbb{E}^*\left[L^*\left(z_i^*; h\right)\right] = 0$ so that $\mathbb{E}^*\left[\hat{L}_n\left(z_i^*; h\right)\right] = 0$. Thus,

$$\mathbb{V}^*\left[L^*\left(z_i^*; h\right)\right] = \left(\frac{n-1}{n}\right)^2 \mathbb{V}^*\left[\hat{L}_n\left(z_i^*; h\right)\right] = \left(\frac{n-1}{n}\right)^2 \sum_{i=1}^{n} \hat{L}_n\left(z_i; h\right)\hat{L}_n\left(z_i; h\right)' n^{-1} = \left(\frac{n-1}{n}\right)^2 \hat{\Sigma}_n(h).$$

We may now rewrite,

$$\begin{aligned}
W^*\left(z_i^*, z_j^*; h\right) &= U\left(z_i^*, z_j^*; h\right) - \frac{1}{2}\left(L^*\left(z_i^*; h\right) + L^*\left(z_j^*; h\right)\right) - \theta^*(h) \\
&= U\left(z_i^*, z_j^*; h\right) - \frac{1}{2}\left(\frac{n-1}{n}\right)\left(\hat{L}_n\left(z_i^*; h\right) + \hat{L}_n\left(z_j^*; h\right)\right) - \theta^*(h).
\end{aligned}$$

Thus,

$$
\begin{aligned}
\mathbb{V}^*\left[W^*\left(z_i^*, z_j^*; h\right)\right] &= \mathbb{V}^*\left[U\left(z_i^*, z_j^*; h\right)\right] \\
&\quad + \frac{1}{4}\left(\frac{n-1}{n}\right)^2 \mathbb{V}^*\left[\hat{L}_n\left(z_i^*; h\right) + \hat{L}_n\left(z_j^*; h\right)\right] \\
&\quad - \frac{1}{2}\left(\frac{n-1}{n}\right)\mathbb{C}^*\left[U\left(z_i^*, z_j^*; h\right), \hat{L}_n\left(z_i^*; h\right)\right] - \frac{1}{2}\left(\frac{n-1}{n}\right)\mathbb{C}^*\left[U\left(z_i^*, z_j^*; h\right), \hat{L}_n\left(z_j^*; h\right)\right] \\
&\quad - \frac{1}{2}\left(\frac{n-1}{n}\right)\mathbb{C}^*\left[\hat{L}_n\left(z_i^*; h\right), U\left(z_i^*, z_j^*; h\right)\right] - \frac{1}{2}\left(\frac{n-1}{n}\right)\mathbb{C}^*\left[\hat{L}_n\left(z_j^*; h\right), U\left(z_i^*, z_j^*; h\right)\right].
\end{aligned}
$$

The first term is,

$$
\begin{aligned}
\mathbb{V}^*\left[U\left(z_i^*, z_j^*; h\right)\right] &= \mathbb{E}^*\left[U\left(z_i^*, z_j^*; h\right) U\left(z_i^*, z_j^*; h\right)'\right] - \theta^*(h)\,\theta^*(h)' \\
&= \sum_{i=1}^n \sum_{j=1}^n U\left(z_i, z_j; h\right) U\left(z_i, z_j; h\right)' n^{-2} - \left(\frac{n-1}{n}\right)^2 \hat{\theta}_n(h)\,\hat{\theta}_n(h)' \\
&= h^{-(d+2)}\left[n^{-1}(n-1)\,\hat{\Delta}_{3,n}(h) - \left(\frac{n-1}{n}\right)^2\left(\hat{\Delta}_{3,n}(h) - \hat{\Delta}_{2,n}(h)\right)\right] \\
&= h^{-(d+2)}\left(\frac{n-1}{n}\right)^2\left[n(n-1)^{-1}\,\hat{\Delta}_{3,n}(h) - \left(\hat{\Delta}_{3,n}(h) - \hat{\Delta}_{2,n}(h)\right)\right] \\
&= h^{-(d+2)}\left(\frac{n-1}{n}\right)^2\left[\hat{\Delta}_{2,n}(h) + (n-1)^{-1}\,\hat{\Delta}_{3,n}(h)\right].
\end{aligned}
$$

The second term is,

$$
\frac{1}{4}\left(\frac{n-1}{n}\right)^2 \mathbb{V}^*\left[\hat{L}_n\left(z_i^*; h\right) + \hat{L}_n\left(z_j^*; h\right)\right] = \frac{1}{2}\left(\frac{n-1}{n}\right)^2 \hat{\Sigma}_n(h).
$$

For the third and fourth terms recall that $\mathbb{E}^*\left[\hat{L}_n\left(z_i^*; h\right)\right] = 0$. Thus,

$$
\begin{aligned}
\mathbb{C}^*\left[U\left(z_i^*, z_j^*; h\right), \hat{L}_n\left(z_i^*; h\right)\right] &= \sum_{i=1}^n \sum_{j=1}^n \left(U\left(z_i, z_j; h\right) - \frac{n-1}{n}\hat{\theta}_n(h)\right)\hat{L}_n\left(z_i; h\right)' n^{-2} \\
&= n^{-2}\sum_{i=1}^n\left[\sum_{j=1}^n\left\{U\left(z_i, z_j; h\right) - \frac{n-1}{n}\hat{\theta}_n(h)\right\}\right]\hat{L}_n\left(z_i; h\right)' \\
&= n^{-2}\sum_{i=1}^n\left[\sum_{j=1}^n U\left(z_i, z_j; h\right) - (n-1)\,\hat{\theta}_n(h)\right]\hat{L}_n\left(z_i; h\right)' \\
&= n^{-2}(n-1)\sum_{i=1}^n\left[(n-1)^{-1}\sum_{j=1}^n U\left(z_i, z_j; h\right) - \hat{\theta}_n(h)\right]\hat{L}_n\left(z_i; h\right)' \\
&= \left(\frac{n-1}{n}\right) n^{-1}\sum_{i=1}^n \hat{L}_n\left(z_i; h\right)\hat{L}_n\left(z_i; h\right)' \\
&= \left(\frac{n-1}{n}\right)\hat{\Sigma}_n(h).
\end{aligned}
$$

Putting this all together yields,

$$
\begin{aligned}
\mathbb{V}^*\left[W^*\left(z_i^*, z_j^*; h\right)\right] &= \left(\frac{n-1}{n}\right)^2 h^{-(d+2)}\left(\hat{\Delta}_{2,n}(h) + (n-1)^{-1}\hat{\Delta}_{3,n}(h)\right) \\
&\quad + \frac{1}{2}\left(\frac{n-1}{n}\right)^2 \hat{\Sigma}_n(h) \\
&\quad - \left(\frac{n-1}{n}\right)^2 \hat{\Sigma}_n(h) - \left(\frac{n-1}{n}\right)^2 \hat{\Sigma}_n(h) \\
&= \left(\frac{n-1}{n}\right)^2 h^{-(d+2)}\left(\hat{\Delta}_{2,n}(h) + (n-1)^{-1}\hat{\Delta}_{3,n}(h)\right) - \frac{3}{2}\left(\frac{n-1}{n}\right)^2 \hat{\Sigma}_n(h).
\end{aligned}
$$

Recall from Cattaneo, Crump and Jansson (2011) that if $n^2h^d \to \infty$,

$$
\hat{\Sigma}_n(h) = \Sigma + 2n\binom{n}{2}^{-1} h^{-(d+2)}\Delta + o_p\left(1 + n^{-1}h^{-(d+2)}\right), \qquad \hat{\Delta}_n(h) = \Delta + o_p(1).
$$

Thus, applying these results to the (conditional) Hoeffding decomposition yields,

$$
\begin{aligned}
\mathbb{V}[\hat{\theta}^*_m(h)] &= \frac{1}{m}\mathbb{V}\left[L^*\left(z_i^*; h\right)\right] + \binom{m}{2}^{-1}\mathbb{V}\left[W^*\left(z_i^*, z_j^*; h\right)\right] \\
&= m^{-1}\left(\frac{n-1}{n}\right)^2\hat{\Sigma}_n(h) + \binom{m}{2}^{-1}\left(\frac{n-1}{n}\right)^2 h^{-(d+2)}\left(\hat{\Delta}_{2,n}(h) + (n-1)^{-1}\hat{\Delta}_{3,n}(h)\right) \\
&\quad - \binom{m}{2}^{-1}\frac{3}{2}\left(\frac{n-1}{n}\right)^2\hat{\Sigma}_n(h) \\
&= m^{-1}\left(\frac{n-1}{n}\right)^2\left[\Sigma + 2n\binom{n}{2}^{-1}h^{-(d+2)}\Delta + o_p\left(1 + n^{-1}h^{-(d+2)}\right)\right] \\
&\quad + \binom{m}{2}^{-1}\left(\frac{n-1}{n}\right)^2 h^{-(d+2)}\left[\Delta + o_p(1)\right] \\
&\quad + \binom{m}{2}^{-1}\left(\frac{n-1}{n}\right)^2 h^{-(d+2)}(n-1)^{-1}\left[\Delta + o_p(1)\right] \\
&\quad - \frac{3}{2}\binom{m}{2}^{-1}\left(\frac{n-1}{n}\right)^2\left[\Sigma + 2n\binom{n}{2}^{-1}h^{-(d+2)}\Delta + o_p\left(1 + n^{-1}h^{-(d+2)}\right)\right] \\
&= m^{-1}\Sigma + 2m^{-1}n\binom{n}{2}^{-1}h^{-(d+2)}\Delta + o_p\left(m^{-1} + m^{-1}n^{-1}h^{-(d+2)}\right) \\
&\quad + \binom{m}{2}^{-1}h^{-(d+2)}\left[\Delta + o_p(1)\right] \\
&= m^{-1}\Sigma + 2m^{-1}n\binom{n}{2}^{-1}h^{-(d+2)}\Delta + \binom{m}{2}^{-1}h^{-(d+2)}\Delta + o_p\left(m^{-1} + m^{-1}n^{-1}h^{-(d+2)}\right),
\end{aligned}
$$

However,

$$
\frac{n}{m}\binom{n}{2}^{-1} = \frac{n}{m}\binom{m}{2}\binom{m}{2}^{-1}\binom{n}{2}^{-1} = \frac{n}{m}\frac{m(m-1)}{2}\frac{2}{n(n-1)}\binom{m}{2}^{-1} = \frac{m-1}{n-1}\binom{m}{2}^{-1}.
$$

Thus,

$$\mathbb{V}[\hat{\theta}^*_m(h)] = m^{-1}\Sigma + 2\frac{m}{n}\binom{m}{2}^{-1} h^{-(d+2)}\Delta + \binom{m}{2}^{-1} h^{-(d+2)}\Delta + o_p\left(m^{-1} + n^{-1} + mn^{-2} + m^{-1}n^{-1}h^{-(d+2)}\right),$$

if $n^2h^d \to \infty$.

4.2 (Conditional) Moment Bounds

4.2.1 Term: $\mathbb{E}^*[(\lambda' U^*_{h,ij})^s]$

Lemma 4.2.1:

$$\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^s\right] \leq C\tilde{T}^{(s)}_{1,n}(\lambda;h)$$

Proof of Lemma 4.2.1:

$$\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^s\right] = n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\lambda' U_{h,ij}\right)^s = n^{-2}\sum_{i=1}^{n}\sum_{j=1,i\neq 1}^{n}\left(\lambda' U_{h,ij}\right)^s = \left(1-n^{-1}\right)\tilde{T}^{(s)}_{1,n}(\lambda;h),$$

which gives the result. ■

4.2.2 Term: $\mathbb{E}^*[(\mathbb{E}^*[(\lambda' U^*_{h,ij})^s|z^*_i])^2]$

Lemma 4.2.2:

$$\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left.\left(\lambda' U^*_{h,ij}\right)^s\right| z^*_i\right]\right)^2\right] \leq Cn^{-1}\tilde{T}^{(2s)}_{1,n}(\lambda;h) + C\tilde{T}^{(s)}_{2,n}(\lambda;h)$$

Proof of Lemma 4.2.2:

$$\begin{aligned}
\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left.\left(\lambda' U^*_{h,ij}\right)^s\right| z^*_i\right]\right)^2\right] &= n^{-1}\sum_{i=1}^{n}\left(n^{-1}\sum_{j=1}^{n}\left(\lambda' U_{h,ij}\right)^s\right)^2 \\
&= n^{-3}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^{2s} + n^{-3}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)^s\left(\lambda' U_{h,ik}\right)^s \\
&= n^{-1}\left(1-n^{-1}\right)\tilde{T}^{(2s)}_{1,n}(\lambda;h) + \left(1-n^{-1}\right)\left(1-2n^{-1}\right)\tilde{T}^{(s)}_{2,n}(\lambda;h),
\end{aligned}$$

which gives the result. ■

4.2.3 Term: $\mathbb{E}^*[(\mathbb{E}^*[\lambda' U^*_{h,ij}|z^*_i])^4]$

Lemma 4.2.3:

$$\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left.\lambda' U^*_{h,ij}\right| z^*_i\right]\right)^4\right] \leq Cn^{-3}\tilde{T}^{(4)}_{1,n}(\lambda;h) + Cn^{-2}\tilde{T}^{(2)}_{2,n}(\lambda;h) + Cn^{-1}\tilde{T}_{4,n}(\lambda;h) + C\tilde{T}_{7,n}(\lambda;h)$$

Proof of Lemma 4.2.3:

$$
\begin{aligned}
\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U_{h,ij}^* \middle| z_i^*\right]\right)^4\right] &= n^{-1}\sum_{i=1}^{n}\left(n^{-1}\sum_{j=1}^{n}\lambda' U_{h,ij}\right)^4 \\
&= n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\lambda' U_{h,ij}\right)^2+\sum_{j=1}^{n}\sum_{k=1,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)\right)^2 \\
&\leq 2n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\lambda' U_{h,ij}\right)^2\right)^2+2n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\sum_{k=1,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)\right)^2,
\end{aligned}
$$

where

$$
2n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\lambda' U_{h,ij}\right)^2\right)^2 = 2n^{-5}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\lambda' U_{h,ij}\right)^4+2n^{-5}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,ik}\right)^2,
$$

and

$$
\begin{aligned}
& n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\sum_{k=1,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)\right)^2 \\
= \;& 2n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,ik}\right)^2 \\
& +4n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq j,l\neq k}^{n}\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,il}\right) \\
& +n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq j,l\neq k}^{n}\sum_{r=1,r\neq i,r\neq j,r\neq k,r\neq l}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,ir}\right).
\end{aligned}
$$

which gives the result. ■

4.2.4 Term: $\mathbb{E}^*[(\mathbb{E}^*[(\lambda' U_{h,ij}^*)(\lambda' U_{h,jk}^*)|z_i^*])^2]$

Lemma 4.2.4:

$$
\begin{aligned}
& \mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U_{h,ij}^*\right)\left(\lambda' U_{h,jk}^*\right)\middle| z_i^*\right]\right)^2\right] \\
\leq \;& Cn^{-3}\tilde{T}_{1,n}^{(4)}(\lambda;h)+Cn^{-2}\tilde{T}_{2,n}^{(2)}(\lambda;h)+Cn^{-2}\tilde{T}_{9,n}(\lambda;h) \\
& +Cn^{-1}\tilde{T}_{4,n}(\lambda;h)+Cn^{-1}\tilde{T}_{5,n}(\lambda;h)+Cn^{-1}\tilde{T}_{6,n}(\lambda;h)+C\tilde{T}_{8,n}(\lambda;h)
\end{aligned}
$$

Proof of Lemma 4.2.4:

$$
\begin{aligned}
& \mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z^*_i\right]\right)^2\right] \\
= \;& n^{-1}\sum_{i=1}^{n}\left(n^{-2}\sum_{j=1}^{n}\left(\lambda' U_{h,ij}\right)\sum_{k=1}^{n}\left(\lambda' U_{h,jk}\right)\right)^2 \\
= \;& n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2+\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)\right)^2 \\
\leq \;& 2n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2\right)^2+2n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)\right)^2,
\end{aligned}
$$

where

$$
n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^2\right)^2 = n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left(\lambda' U_{h,ij}\right)^4+n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,ik}\right)^2,
$$

and

$$
\begin{aligned}
& n^{-5}\sum_{i=1}^{n}\left(\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)\right)^2 \\
&= n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,jk}\right)^2+n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)^2\left(\lambda' U_{h,ik}\right) \\
&\quad + n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq j,l\neq k}^{n}\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,jl}\right) \\
&\quad + 2n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq k}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,kl}\right) \\
&\quad + n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq j,l\neq k}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,lk}\right) \\
&\quad + n^{-5}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq j,l\neq k}^{n}\sum_{r=1,r\neq i,r\neq j,r\neq k,r\neq l}^{n}\left(\lambda' U_{h,ij}\right)\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,lr}\right)
\end{aligned}
$$

which gives the result. ■

4.2.5 Term: $\mathbb{E}^*[(\lambda' U^*_{h,ij})^2(\mathbb{E}[(\lambda' U^*_{h,jk})|z^*_j])^2]$

Lemma 4.2.5:

$$
\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^2\left(\mathbb{E}\left[\left(\lambda' U^*_{h,jk}\right)\middle| z^*_j\right]\right)^2\right] \leq Cn^{-2}\tilde{T}^{(4)}_{1,n}\left(\lambda;h\right)+Cn^{-1}\tilde{T}^{(2)}_{2,n}\left(\lambda;h\right)+C\tilde{T}_{4,n}\left(\lambda;h\right)
$$

Proof of Lemma 4.2.5:

$$\begin{aligned}
\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^2\left(\mathbb{E}\left[\left(\lambda' U^*_{h,jk}\right)\middle| z^*_j\right]\right)^2\right] &= n^{-2}\sum_{i=1}^n\sum_{j=1}^n\left(\lambda' U_{h,ij}\right)^2\left(n^{-1}\sum_{k=1}^n \lambda' U_{h,jk}\right)^2 \\
&= n^{-4}\sum_{i=1}^n\sum_{j=1,j\neq i}^n\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,ji}+\sum_{k=1,k\neq i,k\neq j}^n \lambda' U_{h,jk}\right)^2 \\
&\leq 2n^{-4}\sum_{i=1}^n\sum_{j=1,j\neq i}^n\left(\lambda' U_{h,ij}\right)^4 \\
&\quad +2n^{-4}\sum_{i=1}^n\sum_{j=1,j\neq i}^n\sum_{k=1,k\neq i,k\neq j}^n\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,jk}\right)^2 \\
&\quad +2n^{-4}\sum_{i=1}^n\sum_{j=1,j\neq i}^n\sum_{k=1,k\neq i,k\neq j}^n\sum_{l=1,l\neq i,l\neq j,l\neq k}^n\left(\lambda' U_{h,ij}\right)^2\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,jl}\right)
\end{aligned}$$

4.2.6 Term: $\mathbb{E}^*[(\mathbb{E}^*[\mathbb{E}^*[\lambda' U^*_{h,ij}|z^*_i](\lambda' U^*_{h,ij})|z^*_j])^2]$

Lemma 4.2.6:

$$\begin{aligned}
&\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\mathbb{E}^*\left[\lambda' U^*_{h,ij}\middle| z^*_i\right]\left(\lambda' U^*_{h,ij}\right)\middle| z^*_j\right]\right)^2\right] \\
\leq\ & Cn^{-3}\tilde{T}^{(4)}_{1,n}(\lambda;h)+Cn^{-2}\tilde{T}^{(2)}_{2,n}(\lambda;h)+Cn^{-2}\tilde{T}_{9,n}(\lambda;h) \\
&+Cn^{-1}\tilde{T}_{4,n}(\lambda;h)+Cn^{-1}\tilde{T}_{5,n}(\lambda;h)+Cn^{-1}\tilde{T}_{6,n}(\lambda;h)+C\tilde{T}_{8,n}(\lambda;h)
\end{aligned}$$

Proof of Lemma 4.2.6:

$$\begin{aligned}
\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\mathbb{E}^*\left[\lambda' U^*_{h,ij}\middle| z^*_i\right]\left(\lambda' U^*_{h,ij}\right)\middle| z^*_j\right]\right)^2\right] &= n^{-1}\sum_{i=1}^n\left(n^{-1}\sum_{j=1}^n\left(n^{-1}\sum_{k=1}^n\lambda' U_{h,kj}\right)\left(\lambda' U_{h,ij}\right)\right)^2 \\
&= \mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z^*_i\right]\right)^2\right]
\end{aligned}$$

4.2.7 Term: $\mathbb{E}^*[(\mathbb{E}^*[(\lambda' U^*_{h,ik})(\lambda' U^*_{h,jk})|z^*_i,z^*_j])^2]$

Lemma 4.2.7:

$$\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z^*_i,z^*_j\right]\right)^2\right]\leq Cn^{-1}\tilde{T}^{(4)}_{1,n}(\lambda;h)+Cn^{-1}\tilde{T}^{(2)}_{2,n}(\lambda;h)+C\tilde{T}_{6,n}(\lambda;h)$$

Proof of Lemma 4.2.7:

$$
\begin{aligned}
& \mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z^*_i, z^*_j\right]\right)^2\right] \\
= \;& n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{n}\sum_{k=1}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)\right)^2 \\
= \;& n^{-4}\sum_{i=1}^{n}\sum_{k=1,k\neq i}^{n}\left(\lambda' U_{h,ik}\right)^4 + 2n^{-4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\left(\lambda' U_{h,ik}\right)^2\left(\lambda' U_{h,jk}\right)^2 \\
& + n^{-4}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1,k\neq i,k\neq j}^{n}\sum_{l=1,l\neq i,l\neq j,l\neq k}^{n}\left(\lambda' U_{h,ik}\right)\left(\lambda' U_{h,jk}\right)\left(\lambda' U_{h,il}\right)\left(\lambda' U_{h,jl}\right)
\end{aligned}
$$

4.3 Expansions and Convergence in Probability of Bootstrap m-varying U-statistics

For $\lambda \in \Lambda$ define the following m-varying (bootstrap) U-statistics,

$$
\tilde{T}^{(s)*}_{1,m}(\lambda; h) = \binom{m}{2}^{-1}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left(\lambda' U^*_{h,ij}\right)^s
$$

$$
\begin{aligned}
\tilde{T}^{(1)*}_{2,m}(\lambda; h) &= \binom{m}{3}^{-1}\sum_{i=1}^{m-2}\sum_{j=i+1}^{m-1}\sum_{k=j+1}^{m} U^*_{h,ijk}(\lambda), \\
& U^*_{h,ijk}(\lambda) = \frac{\left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,ik}\right) + \left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,jk}\right) + \left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jk}\right)}{3},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{T}^{(1)*}_{3,m}(\lambda; h) &= \binom{m}{4}^{-1}\sum_{i=1}^{m-3}\sum_{j=i+1}^{m-2}\sum_{k=j+1}^{m-1}\sum_{l=k+1}^{m} U^*_{h,ijkl}(\lambda), \\
& U^*_{h,ijkl}(\lambda) = \frac{\left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,kl}\right) + \left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jl}\right) + \left(\lambda' U^*_{h,il}\right)\left(\lambda' U^*_{h,jk}\right)}{3}.
\end{aligned}
$$

As in the non-bootstrap case, these m-varying (bootstrap) U-statistics are the building blocks of $\hat{\Sigma}^*_m(h)$, $\hat{\Delta}^*_m(h)$, $\hat{\Delta}^*_{2,m}(h)$, and $\hat{\Delta}^*_{3,m}(h)$.

4.3.1 Term: $\hat{\Sigma}^*_m(h)$

Using Lemma 3.1.1,

$$
\frac{1}{m}\lambda'\hat{\Sigma}^*_m(h)\lambda = 2\binom{m}{2}^{-1}\tilde{T}^{(2)*}_{1,m}(\lambda; h) + \frac{4}{m}\frac{m-2}{m-1}\tilde{T}^{(1)*}_{2,m}(\lambda; h) - \frac{4}{m}\left(\tilde{T}^{(1)*}_{1,m}(\lambda; h)\right)^2,
$$

4.3.2 Term: $\hat{\Delta}_m^*(h)$

Using Lemma 3.1.2,

$$\begin{aligned}\lambda'\hat{\Delta}_m^*(h)\lambda &= \left(1-\frac{2}{m-1}+\frac{2}{(m-1)^2}\right)h^{d+2}\tilde{T}_{1,m}^{(2)*}(\lambda;h)-\left(\frac{2(m-2)}{m-1}-\frac{6(m-2)}{(m-1)^2}\right)h^{d+2}\tilde{T}_{2,m}^{(1)*}(\lambda;h)\\&\quad+\frac{2(m-2)(m-3)}{(m-1)^2}h^{d+2}\tilde{T}_{3,m}^{(1)*}(\lambda;h)-h^{d+2}\left(\tilde{T}_{1,m}^{(1)*}(\lambda;h)\right)^2,\end{aligned}$$

$$\lambda'\hat{\Delta}_{2,m}^*(h)\lambda = h^{d+2}\tilde{T}_{1,m}^{(2)*}(\lambda;h)-h^{d+2}\left(\tilde{T}_{1,m}^{(1)*}(\lambda;h)\right)^2,$$

$$\lambda'\hat{\Delta}_{3,m}^*(h)\lambda = h^{d+2}\tilde{T}_{1,m}^{(2)*}(\lambda;h).$$

4.3.3 Term: $\tilde{T}_{1,m}^{(s)*}(\lambda;h)$

Lemma 4.3.3:

$$\tilde{T}_{1,m}^{(s)*}(\lambda;h)=\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\right]+O_p\left(m^{-1/2}h^{-(s-1)d-s\mathbf{1}(s>1)}+m^{-1}h^{-(2s-1)d/2-s}\right).$$

Proof of Lemma 4.3.3: Using the Hoeffding decomposition,

$$\tilde{T}_{1,m}^{(s)*}(\lambda;h)=\binom{m}{2}^{-1}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(\lambda'U_{h,ij}^*)^s=\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\right]+\tilde{T}_{11,m}^{(s)*}(\lambda;h)+\tilde{T}_{12,m}^{(s)*}(\lambda;h),$$

where

$$\tilde{T}_{11,m}^{(s)*}(\lambda;h)=\frac{1}{m}\sum_{i=1}^{m}2\left(\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\middle|z_i^*\right]-\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\right]\right),$$

$$\tilde{T}_{12,m}^{(s)*}(\lambda;h)=\binom{m}{2}^{-1}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left((\lambda'U_{h,ij}^*)^s-\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\middle|z_i^*\right]-\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\middle|z_j^*\right]+\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\right]\right).$$

Now,

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{T}_{11,m}^{(s)*}(\lambda;h)\right)^2\right] &= \frac{4}{m}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\middle|z_i^*\right]-\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\right]\right)^2\right]\right]\leq\frac{4}{m}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\middle|z_i^*\right]\right)^2\right]\right]\\&\leq \frac{C}{m}\frac{1}{n}\mathbb{E}\left[\tilde{T}_{1,n}^{(2s)}(\lambda;h)\right]+\frac{C}{m}\mathbb{E}\left[\tilde{T}_{2,n}^{(s)}(\lambda;h)\right]\\&= O\left(m^{-1}n^{-1}h^{-(2s-1)d-2s}+m^{-1}h^{-2(s-1)d-2s\mathbf{1}(s>1)}\right),\end{aligned}$$

by Lemma 4.2.2, Lemma 3.2.1 and Lemma 3.2.2, and

$$\begin{aligned}\mathbb{E}\left[\left(T_{12,m}^{(s)*}(\lambda;h)\right)^2\right] &= \binom{m}{2}^{-1}\mathbb{E}\left[\mathbb{E}^*\left[\left((\lambda'U_{h,ij}^*)^s-\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s|z_i^*\right]-\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s|z_j^*\right]+\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^s\right]\right)^2\right]\right]\\&\leq \binom{m}{2}^{-1}\mathbb{E}\left[\mathbb{E}^*\left[(\lambda'U_{h,ij}^*)^{2s}\right]\right]\leq Cm^{-2}\mathbb{E}\left[\tilde{T}_{1,n}^{(2s)}(\lambda;h)\right]\\&= O\left(m^{-2}h^{-(2s-1)d-2s}\right),\end{aligned}$$

by Lemma 4.2.1 and Lemma 3.2.1. This gives the result. ■

4.3.4 Term: $T_{2,m}^{(1)*}(\lambda; h)$

Lemma 4.3.4:

$$\tilde{T}_{2,m}^{(1)*}(\lambda; h) = \mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U_{h,ij}^* | z_i^*\right]\right)^2\right] + O_p\left(m^{-1/2} + m^{-1}h^{-d/2-2} + m^{-3/2}h^{-d-2} + m^{-2}h^{-3d/2-2}\right).$$

Proof of Lemma 4.3.4: Recall that

$$\tilde{T}_{2,m}^{(1)*}(\lambda; h) = \binom{m}{3}^{-1} \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m} U_{h,ijk}^{(1)*}(\lambda),$$

$$U_{h,ijk}^{(1)*}(\lambda) = \frac{\left(\lambda' U_{h,ij}^*\right)\left(\lambda' U_{h,ik}^*\right) + \left(\lambda' U_{h,ij}^*\right)\left(\lambda' U_{h,jk}^*\right) + \left(\lambda' U_{h,ik}^*\right)\left(\lambda' U_{h,jk}^*\right)}{3}.$$

We drop the superscript to save notation. Using the Hoeffding decomposition,

$$\tilde{T}_{2,m}^*(\lambda; h) = \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda)\right] + \tilde{T}_{21,m}^*(\lambda; h) + \tilde{T}_{22,m}^*(\lambda; h) + \tilde{T}_{23,m}^*(\lambda; h),$$

where

$$\mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda)\right] = \mathbb{E}^*\left[\left(\lambda' U_{h,ij}^*\right)\left(\lambda' U_{h,ik}^*\right)\right] = \mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U_{h,ij}^* \middle| z_i^*\right]\right)^2\right],$$

$$\tilde{T}_{21,m}^*(\lambda; h) = \frac{1}{m}\sum_{i=1}^{m} 3\left(\mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_i^*\right] - \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda)\right]\right),$$

$$\begin{aligned} &\tilde{T}_{22,m}^*(\lambda; h) \\ = &\binom{m}{2}^{-1} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} 3\left(\mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_i^*, z_j^*\right] - \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_i^*\right] - \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| |z_j^*\right] + \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) |\right]\right), \end{aligned}$$

$$\begin{aligned} &\tilde{T}_{23,m}^*(\lambda; h) \\ = &\binom{m}{3}^{-1} \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \sum_{k=j+1}^{m} (\lambda' U_{h,ijk}^*\left(^l\right) + \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_i^*\right] + \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_j^*\right] + \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_k^*\right] \\ &-\mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_i^*, z_j^*\right] - \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_i^*, z_k^*\right] - \mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_j^*, z_k^*\right] \\ &-\mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda)\right]). \end{aligned}$$

Now,

$$\mathbb{E}^*\left[\lambda' U_{h,ijk}^*(\lambda) \middle| z_i^*\right] = \frac{1}{3}\left(\mathbb{E}^*\left[\lambda' U_{h,ij}^* \middle| z_i^*\right]\right)^2 + \frac{2}{3}\mathbb{E}^*\left[\left(\lambda' U_{h,ij}^*\right)\left(\lambda' U_{h,jk}^*\right) \middle| z_i^*\right],$$

and thus

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}^*\left[\left(\tilde{T}^*_{21,m}(\lambda;h)\right)^2\right]\right] &= \frac{9}{m}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ijk}(\lambda)\middle| z^*_i\right] - \mathbb{E}^*\left[\lambda' U^*_{h,ijk}(\lambda)\right]\right)^2\right]\right] \\
&\leq Cm^{-1}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ijk}(\lambda)\middle| z^*_i\right]\right)^2\right]\right] \\
&\leq Cm^{-1}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ij}\middle| z^*_i\right]\right)^4\right]\right] + Cm^{-1}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z^*_i\right]\right)^2\right]\right] \\
&= O\left(m^{-1}n^{-3}h^{-3d-4} + m^{-1}n^{-2}h^{-2d-4} + m^{-1}n^{-1}h^{-d-4} + m^{-1}\right),
\end{aligned}$$

because

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ij}\middle| z^*_i\right]\right)^4\right]\right] &\leq \frac{C}{n^3}\mathbb{E}\left[\tilde{T}^{(4)}_{1,n}(\lambda;h)\right] + \frac{C}{n^2}\mathbb{E}\left[\tilde{T}^{(2)}_{2,n}(\lambda;h)\right] + \frac{C}{n}\mathbb{E}\left[\tilde{T}_{4,n}(\lambda;h)\right] + C\mathbb{E}\left[\tilde{T}_{7,n}(\lambda;h)\right] \\
&= Cn^{-3}h^{-3d-4} + Cn^{-2}h^{-2d-4} + Cn^{-1}h^{-d-2} + C,
\end{aligned}$$

by Lemma 4.2.3, Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.4 and Lemma 3.2.7 and

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,jk}\right)|z^*_i\right]\right)^2\right]\right] \\
\leq\ & \frac{1}{n^3}\mathbb{E}\left[\tilde{T}^{(4)}_{1,n}(\lambda;h)\right] + \frac{C}{n^2}\mathbb{E}\left[\tilde{T}^{(2)}_{2,n}(\lambda;h)\right] + \frac{C}{n^2}\mathbb{E}\left[\tilde{T}_{9,n}(\lambda;h)\right] + \frac{C}{n}\mathbb{E}\left[\tilde{T}_{4,n}(\lambda;h)\right] \\
&\quad + \frac{C}{n}\mathbb{E}\left[\tilde{T}_{5,n}(\lambda;h)\right] + \frac{C}{n}\mathbb{E}\left[\tilde{T}_{6,n}(\lambda;h)\right] + C\mathbb{E}\left[\tilde{T}_{8,n}(\lambda;h)\right] \\
=\ & Cn^{-3}h^{-3d-4} + Cn^{-2}h^{-2d-4} + Cn^{-1}h^{-d-2} \\
&\quad + Cn^{-1}h^{-d-3} + Cn^{-1}h^{-d-4} + C,
\end{aligned}$$

by Lemma 4.2.4, Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.9, Lemma 3.2.4, Lemma 3.2.5, Lemma 3.2.6 and Lemma 3.2.8.

Similarly,

$$\mathbb{E}^*\left[\lambda' U^*_{h,ijk}(\lambda)\middle| z^*_i, z^*_j\right] = \frac{1}{3}\left(\lambda' U^*_{h,ij}\right)\mathbb{E}^*\left[\lambda' U^*_{h,ik}\middle| z^*_i\right] + \frac{1}{3}\left(\lambda' U^*_{h,ij}\right)\mathbb{E}^*\left[\lambda' U^*_{h,jk}\middle| z^*_j\right] + \frac{1}{3}\mathbb{E}^*\left[\left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z^*_i, z^*_j\right],$$

and hence

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{E}^*\left[\left(\tilde{T}^*_{22,m}(\lambda;h)\right)^2\right]\right] \\
=\ & 9\binom{m}{2}^{-1}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ijk}(\lambda)\middle| z^*_i, z^*_j\right] - \mathbb{E}^*\left[\lambda' U^*_{h,ijk}(\lambda)\middle| z^*_i\right] - \mathbb{E}^*\left[\lambda' U^*_{h,ijk}(\lambda)\middle| z^*_j\right] + \mathbb{E}^*\left[\lambda' U^*_{h,ijk}(\lambda)\right]\right)^2\right]\right] \\
\leq\ & Cm^{-2}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ijk}(\lambda)\middle| z^*_i, z^*_j\right]\right)^2\right]\right] \\
\leq\ & Cm^{-2}\mathbb{E}\left[\mathbb{E}^*\left[\left(\left(\lambda' U^*_{h,ij}\right)\mathbb{E}^*\left[\lambda' U^*_{h,ik}\middle| z^*_i\right]\right)^2\right]\right] + Cm^{-2}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z^*_i, z^*_j\right]\right)^2\right]\right] \\
=\ & O\left(m^{-2}n^{-2}h^{-3d-4} + m^{-2}n^{-1}h^{-2d-4} + m^{-2}h^{-d-4}\right),
\end{aligned}$$

because

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^2\left(\mathbb{E}\left[\lambda' U^*_{h,ik}\middle| z^*_i\right]\right)^2\right]\right] &\leq \frac{C}{n^2}\mathbb{E}\left[\tilde{T}^{(4)}_{1,n}(\lambda;h)\right] + \frac{C}{n}\mathbb{E}\left[\tilde{T}^{(2)}_{2,n}(\lambda;h)\right] + C\mathbb{E}\left[\tilde{T}_{4,n}(\lambda;h)\right] \\
&= Cn^{-2}h^{-3d-4} + Cn^{-1}h^{-2d-4} + Ch^{-d-2},
\end{aligned}$$

by Lemma 4.2.5, Lemma 3.2.1, Lemma 3.2.2, and Lemma 3.2.4 and

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z^*_i, z^*_j\right]\right)^2\right]\right] &\leq \frac{1}{n^2}\mathbb{E}\left[\tilde{T}^{(4)}_{1,n}(\lambda;h)\right] + \frac{C}{n}\mathbb{E}\left[\tilde{T}^{(2)}_{2,n}(\lambda;h)\right] + C\mathbb{E}\left[\tilde{T}_{6,n}(\lambda;h)\right] \\ &= Cn^{-2}h^{-3d-4} + Cn^{-1}h^{-2d-4} + Ch^{-d-4},\end{aligned}$$

by Lemma 4.2.7, Lemma 3.2.1, Lemma 3.2.2, and Lemma 3.2.6.

Finally,

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}^*\left[\left(\tilde{T}^*_{23,m}(\lambda;h)\right)^2\right]\right] &\leq \binom{m}{3}^{-1}\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' U^*_{h,ijk}(\lambda)\right)^2\right]\right] \leq Cm^{-3}\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^2\left(\lambda' U^*_{h,ik}\right)^2\right]\right] \\ &= Cm^{-3}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^2\middle| z^*_i\right]\right)^2\right]\right] \\ &\leq Cm^{-3}\frac{1}{n}\mathbb{E}\left[\tilde{T}^{(4)}_{1,n}(\lambda;h)\right] + Cm^{-3}\mathbb{E}\left[\tilde{T}^{(2)}_{2,n}(\lambda;h)\right] \\ &= O\left(m^{-3}n^{-1}h^{-3d-4} + m^{-3}h^{-2d-4}\right),\end{aligned}$$

by Lemma 4.2.2, Lemma 3.2.1 and Lemma 3.2.2. This completes the proof. ■

4.3.5 Term: $\tilde{T}^{(1)*}_{3,m}(\lambda;h)$

Lemma 4.3.5:

$$\tilde{T}^{(1)*}_{3,m}(\lambda;h) = \left(\mathbb{E}^*\left[\lambda' U^*_{h,ij}\right]\right)^2 + O_p\left(m^{-1/2} + m^{-1}h^{-3d/2-2}\right).$$

Proof of Lemma 4.3.5: Recall that

$$\tilde{T}^{(1)*}_{3,m}(\lambda;h) = \binom{m}{4}^{-1}\sum_{i=1}^{m-3}\sum_{j=i+1}^{m-2}\sum_{k=j+1}^{m-1}\sum_{l=k+1}^{m} U^{(1)*}_{h,ijkl}(\lambda),$$

$$U^{(1)*}_{ijkl}(\lambda) = \frac{\left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,kl}\right) + \left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jl}\right) + \left(\lambda' U^*_{h,il}\right)\left(\lambda' U^*_{h,jk}\right)}{3}.$$

We drop the superscript to save notation. Using a Hoeffding decomposition,

$$\tilde{T}^*_{3,m}(\lambda;h) = \mathbb{E}^*\left[\lambda' U^*_{h,ijkl}(\lambda)\right] + \tilde{T}^*_{31,m}(\lambda;h) + \tilde{T}^*_{32,m}(\lambda;h) + \tilde{T}^*_{33,m}(\lambda;h) + \tilde{T}^*_{34,m}(\lambda;h),$$

where

$$\tilde{T}^*_{31,m}(\lambda;h) = \frac{1}{m}\sum_{i=1}^{m} 4\left(\mathbb{E}^*\left[\lambda' U_{h,ijkl}(\lambda)\middle| z^*_i\right] - \mathbb{E}^*\left[\lambda' U_{h,ijkl}(\lambda)\right]\right),$$

$$\begin{aligned}&\tilde{T}^*_{32,m}(\lambda;h) \\ &= \binom{m}{2}^{-1}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} 6\left(\mathbb{E}^*\left[\lambda' U_{h,ijkl}(\lambda)\middle| z^*_i, z^*_j\right] - \mathbb{E}^*\left[\lambda' U_{h,ijkl}(\lambda)\middle| z^*_i\right] - \mathbb{E}^*\left[\lambda' U_{h,ijkl}(\lambda)\middle| z^*_j\right] + \mathbb{E}^*\left[\lambda' U_{h,ijkl}(\lambda)\right]\right),\end{aligned}$$

$$\begin{aligned}\tilde{T}^{*}_{33,m} &= \binom{m}{3}^{-1}\sum_{i=1}^{m-2}\sum_{j=i+1}^{m-1}\sum_{k=j+1}^{m} 4(\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}, z^{*}_{j}, z_{k}\right] \\ &+\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}\right] + \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{j}\right] + \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{k}\right] \\ &-\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}, z^{*}_{j}\right] - \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}, z^{*}_{k}\right] - \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{k}, z^{*}_{j}\right] - \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right|\right]),\end{aligned}$$

$$\begin{aligned}\tilde{T}^{*}_{34,m}(\lambda;h) &= \binom{m}{4}^{-1}\sum_{i=1}^{m-3}\sum_{j=i+1}^{m-2}\sum_{k=j+1}^{m-1}\sum_{l=k+1}^{m} (\lambda' U_{h,ijkl}(\lambda)| \\ &-\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}\right] - \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{j}\right] - \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{k}\right] - \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{l}\right] \\ &+\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}, z^{*}_{j}\right] + \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}, z^{*}_{k}\right] + \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}, z^{*}_{l}\right] \\ &+\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{j}, z^{*}_{k}\right] + \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{j}, z^{*}_{l}\right] + \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z_{k}, z^{*}_{l}\right] \\ &-\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}, z^{*}_{j}, z^{*}_{k}\right] - \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}, z^{*}_{j}, z^{*}_{l}\right] \\ &-\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}, z^{*}_{k}, z^{*}_{l}\right] - \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z_{j}, z^{*}_{k}, z^{*}_{l}\right] + \mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right|\right]).\end{aligned}$$

Note that

$$\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right|\right] = \left(\mathbb{E}^{*}\left[\lambda' U^{*}_{h,ij}\right]\right)^{2}, \qquad \mathbb{E}^{*}\left[\lambda' U^{*}_{h,ijkl}|z^{*}_{i}\right] = \mathbb{E}^{*}\left[\lambda' U^{*}_{h,ik}\right| z^{*}_{i}\right]\mathbb{E}^{*}\left[\lambda' U^{*}_{h,jl}\right],$$

and hence

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}^{*}\left[\left(\tilde{T}^{*}_{31,m}(\lambda;h)\right)^{2}\right]\right] &\leq Cm^{-1}\mathbb{E}\left[\mathbb{E}^{*}\left[\left(\mathbb{E}^{*}\left[\lambda' U_{h,ijkl}(\lambda)\right| z^{*}_{i}\right]\right)^{2}\right]\right] \\ &= Cm^{-1}\mathbb{E}\left[\mathbb{E}^{*}\left[\left(\mathbb{E}^{*}\left[\lambda' U^{*}_{h,ik}\right| z^{*}_{i}\right]\right)^{2}\right]\left(\mathbb{E}^{*}\left[\lambda' U^{*}_{h,jl}\right]\right)^{2}\right] \\ &\leq Cm^{-1}\mathbb{E}\left[\left(\mathbb{E}^{*}\left[\left(\mathbb{E}^{*}\left[\lambda' U^{*}_{h,ik}\right| z^{*}_{i}\right]\right)^{2}\right]\right)^{2}\right] \\ &\leq Cm^{-1}\mathbb{E}\left[\left(\frac{1}{n}\tilde{T}^{(2)}_{1,n}(\lambda;h) + \tilde{T}^{(1)}_{2,n}(\lambda;h)\right)^{2}\right] \\ &= O\left(m^{-1} + m^{-3}h^{-2d-4} + m^{-5}h^{-3d-4}\right),\end{aligned}$$

by Lemma 4.2.2 and because

$$\mathbb{E}\left[\left(\tilde{T}^{(2)}_{1,n}(\lambda;h)\right)^{2}\right] = O\left(h^{-2d-4} + n^{-1}h^{-2d-4} + n^{-2}h^{-3d-4}\right),$$

$$\mathbb{E}\left[\left(\tilde{T}^{(1)}_{2,n}(\lambda;h)\right)^{2}\right] = O\left(1 + n^{-1} + n^{-2}h^{-d-4} + n^{-3}h^{-2d-4}\right),$$

by Lemma 3.1.3 and Lemma 3.1.4.

Next, note that

$$\begin{aligned}\mathbb{E}^*\left[\lambda' U^*_{h,ijkl}(\lambda)\middle| z_i^*, z_j^*\right] &= \frac{1}{3}\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,kl}\right)\middle| z_i^*, z_j^*\right] \\ &\quad + \frac{1}{3}\mathbb{E}^*\left[\left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jl}\right)\middle| z_i^*, z_j^*\right] + \frac{1}{3}\mathbb{E}^*\left[\left(\lambda' U^*_{h,il}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z_i^*, z_j^*\right] \\ &= \frac{1}{3}\left(\lambda' U^*_{h,ij}\right)\mathbb{E}^*\left[\lambda' U^*_{h,kl}\right] + \frac{2}{3}\mathbb{E}^*\left[\lambda' U^*_{h,ik}\middle| z_i^*\right]\mathbb{E}^*\left[\lambda' U^*_{h,jl}\middle| z_j^*\right],\end{aligned}$$

and hence

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{T}^*_{32,m}(\lambda;h)\right)^2\right] &\leq C\binom{m}{2}^{-1}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ijkl}(\lambda)\middle| z_i^*, z_j^*\right]\right)^2\right]\right] \\ &\leq Cm^{-2}\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^2\right]\left(\mathbb{E}^*\left[\lambda' U^*_{h,kl}\right]\right)^2\right] + Cm^{-2}\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ik}\middle| z_i^*\right]\right)^2\right]\right)^2\right] \\ &\leq Cm^{-2}\mathbb{E}\left[\tilde{T}^{(4)}_{1,n}(\lambda;h)\right] \\ &= O\left(m^{-2}h^{-3d-4}\right),\end{aligned}$$

by simple bounding arguments, Lemma 4.2.1 and Lemma 3.1.3.

Next, note that

$$\begin{aligned}\mathbb{E}^*\left[\lambda' U^*_{h,ijkl}(\lambda)\middle| z_i^*, z_j^*, z_k^*\right] &= \frac{1}{3}\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)\left(\lambda' U^*_{h,kl}\right)\middle| z_i^*, z_j^*, z_k^*\right] \\ &\quad + \frac{1}{3}\mathbb{E}^*\left[\left(\lambda' U^*_{h,ik}\right)\left(\lambda' U^*_{h,jl}\right)\middle| z_i^*, z_j^*, z_k^*\right] + \frac{1}{3}\mathbb{E}^*\left[\left(\lambda' U^*_{h,il}\right)\left(\lambda' U^*_{h,jk}\right)\middle| z_i^*, z_j^*, z_k^*\right] \\ &= \frac{1}{3}\left(\lambda' U^*_{h,ij}\right)\mathbb{E}^*\left[\lambda' U^*_{h,kl}\middle| z_k^*\right] + \frac{1}{3}\left(\lambda' U^*_{h,ik}\right)\mathbb{E}^*\left[\lambda' U^*_{h,jl}\middle| z_j^*\right] + \frac{1}{3}\left(\lambda' U^*_{h,jk}\right)\mathbb{E}^*\left[\lambda' U^*_{h,il}\middle| z_i^*\right],\end{aligned}$$

and hence

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}^*\left[\left(\tilde{T}^*_{33,m}(\lambda;h)\right)^2\right]\right] &\leq C\binom{m}{3}^{-1}\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ijkl}(\lambda)\middle| z_i^*, z_j^*, z_k^*\right]\right)^2\right]\right] \\ &\leq Cm^{-3}\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^2\left(\mathbb{E}^*\left[\lambda' U^*_{h,kl}\middle| z_k^*\right]\right)^2\right]\right] \\ &\leq Cm^{-3}\mathbb{E}\left[\tilde{T}^{(4)}_{1,n}(\lambda;h)\right] \\ &= O_p\left(m^{-3}h^{-3d-4}\right),\end{aligned}$$

by simple bounding arguments, Lemma 4.2.1 and Lemma 3.1.3.

Finally, note that

$$\left(\lambda' U^*_{h,ijkl}(\lambda)\right)^2 \leq C\left(\lambda' U^*_{h,ij}\right)^2\left(\lambda' U^*_{h,kl}\right)^2 + C\left(\lambda' U^*_{h,ik}\right)^2\left(\lambda' U^*_{h,jl}\right)^2 + C\left(\lambda' U^*_{h,il}\right)^2\left(\lambda' U^*_{h,jk}\right)^2,$$

and hence

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{T}^*_{34,m}(\lambda;h)\right)^2\right] &\leq C\binom{m}{4}^{-1}\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' U^*_{h,ijkl}(\lambda)\right)^2\right]\right] \\ &\leq Cm^{-4}\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^2\right]\right)^2\right] \leq Cm^{-4}\mathbb{E}\left[\tilde{T}^{(4)}_{1,n}(\lambda;h)\right] = O\left(m^{-4}h^{-3d-4}\right),\end{aligned}$$

by simple bounding arguments, Lemma 4.2.1 and Lemma 3.1.3. This completes the proof. ■

4.3.6 Rates of Convergence in Probability of $\hat{\Sigma}^*_m(h)$ and $\hat{\Delta}^*_m(h)$

Now, similarly as in Section 3.1, we can use Lemma 3.1.1 and Lemma 3.1.2 to characterize the rates of convergence in probability of $\hat{\Sigma}^*_{m,n}$ and $\hat{\Delta}^*_{m,n}$. By Lemma 3.1.1,

$$\frac{1}{m}\lambda'\hat{\Sigma}^*_m(h)\lambda = \frac{4}{m}\frac{m-2}{m-1}\tilde{T}^{(1)*}_{2,m}(\lambda;h) - \frac{4}{m}\left(\tilde{T}^{(1)*}_{1,m}(\lambda;h)\right)^2 + 2\binom{m}{2}^{-1}\tilde{T}^{(2)*}_{1,m}(\lambda;h)$$

where

$$\begin{aligned}\tilde{T}^{(1)*}_{1,m}(\lambda;h) &= \mathbb{E}^*\left[\lambda' U^*_{h,ij}\right] + O_p\left(m^{-1/2} + m^{-1}h^{-d/2-1}\right)\\ &= \frac{n-1}{n}\lambda'\hat{\theta}_n(h) + O_p\left(m^{-1/2} + m^{-1}h^{-d/2-1}\right),\end{aligned}$$

$$\begin{aligned}\tilde{T}^{(2)*}_{1,m}(\lambda;h) &= \mathbb{E}^*\left[\left(\lambda' U^*_{h,ij}\right)^2\right] + O_p\left(m^{-1/2}h^{-d-2} + m^{-1}h^{-3d/2-2}\right)\\ &= h^{-(d+2)}\frac{n-1}{n}\lambda'\hat{\Delta}_{3,n}(h)\lambda + O_p\left(m^{-1/2}h^{-d-2} + m^{-1}h^{-3d/2-2}\right),\end{aligned}$$

$$\tilde{T}^{(1)*}_{2,m}(\lambda;h) = \mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ij}|z^*_i\right]\right)^2\right] + O_p\left(m^{-1/2} + m^{-1}h^{-d/2-2} + m^{-3/2}h^{-d-2} + m^{-2}h^{-3d/2-2}\right),$$

by Lemmas 4.3.3 and 4.3.4. Therefore, if $m^2 h^d_m \to \infty$, then

$$\begin{aligned}\frac{1}{m}\lambda'\hat{\Sigma}^*_m(h_m)\lambda &= \frac{1}{m}\left[\lambda'\hat{\Sigma}_n(h_m)\lambda + o_p(1)\right] + 2\binom{m}{2}^{-1}h_m^{-(d+2)}\left[\lambda'\hat{\Delta}_{3,n}(h_m)\lambda + o_p(1)\right]\\ &= \frac{1}{m}\left[\lambda'\Sigma\lambda + o_p(1)\right] + 2\frac{n}{m}\binom{n}{2}^{-1}h_m^{-(d+2)}\left[\lambda'\Delta\lambda + o_p(1)\right] + 2\binom{m}{2}^{-1}h_m^{-(d+2)}\left[\lambda'\Delta\lambda + o_p(1)\right]\\ &= \frac{1}{m}\left[\lambda'\Sigma\lambda + o_p(1)\right] + 2\frac{m}{n}\binom{m}{2}^{-1}h_m^{-(d+2)}\left[\lambda'\Delta\lambda + o_p(1)\right] + 2\binom{m}{2}^{-1}h_m^{-(d+2)}\left[\lambda'\Delta\lambda + o_p(1)\right].\end{aligned}$$

Similarly, by Lemma 4.3.5,

$$\tilde{T}^{(1)*}_{3,m}(\lambda;h) = \left(\mathbb{E}^*\left[\lambda' U^*_{h,ij}\right]\right)^2 + O_p\left(m^{-1/2} + m^{-1}h^{-3d/2-2}\right),$$

and so by Lemma 3.1.2, it follows that

$$\begin{aligned}\lambda'\hat{\Delta}^*_m(h_m)\lambda &= \left(1 - \frac{2}{(m-1)} + \frac{2}{(m-1)^2}\right)h_m^{d+2}\tilde{T}^{*(2)}_{1,m}(\lambda;h) - \left(\frac{2(m-2)}{m-1} - \frac{6(m-2)}{(m-1)^2}\right)h_m^{d+2}\tilde{T}^{*(1)}_{2,m}(\lambda;h)\\ &\quad + \frac{2(m-2)(m-3)}{(m-1)^2}h_m^{d+2}\tilde{T}^{*(1)}_{3,m}(\lambda;h) - h_m^{d+2}\left(\tilde{T}^{*(1)}_{1,m}(\lambda;h)\right)^2\\ &= \lambda'\Delta\lambda + o_p(1), \qquad \text{if } m^2 h^d_m \to \infty.\end{aligned}$$

$$\lambda'\hat{\Delta}^*_{2,m}(h_m)\lambda = h_m^{d+2}\tilde{T}^{*(2)}_{1,m}(\lambda;h) - h_m^{d+2}\left(\tilde{T}^{*(1)}_{1,m}(\lambda;h)\right)^2 = \lambda'\Delta\lambda + o_p(1), \qquad \text{if } m^2 h^d_m \to \infty.$$

$$\lambda'\hat{\Delta}^*_{3,m}\lambda(h_m) = h_m^{d+2}\tilde{T}^{*(2)}_{1,m}(\lambda;h) = \lambda'\Delta\lambda + o_p(1), \qquad \text{if } m^2 h^d_m \to \infty.$$

5 Convergence in Distribution of $\hat{\theta}_m^*(h)$

5.1 Preliminary Lemma

Lemma 5.1: For and $d \in N$, there exist constants C and J (only depending on d) and a collection $l_1, \ldots, l_J \in \Lambda$ such that

$$\sup_{\lambda \in \Lambda} \left(\lambda' M \lambda\right)^2 \leq C \sum_{j=1}^{J} \left(l_j' M l_j\right)^2 .$$

Proof of Lemma 5.1: Let λ be given. For any $d \times d$ matrix M,

$$\lambda' M \lambda = \sum_i \lambda_i^2 M_{ii}^2 + \sum_{i<j} \lambda_i \lambda_j \left(M_{ij} + M_{ji}\right),$$

where M_{ij} denotes element (i,j) of M. Because there are $C_d = d(d+1)/2$ terms on the right-hand side and because $\max_{1 \leq i \leq d} |\lambda_i| \leq 1$, we have:

$$\left(\lambda' M \lambda\right)^2 \leq C_d \left[\sum_i \lambda_i^4 M_{ii}^2 + \sum_{i<j} \lambda_i^2 \lambda_j^2 \left(M_{ij} + M_{ji}\right)^2\right] \leq C_d \left[\sum_i M_{ii}^2 + \sum_{i<j} \left(M_{ij} + M_{ji}\right)^2\right].$$

Now, $M_{ii}^2 = (e_i' M e_i)$, where e_i is the ith unit vector in R^d. Also, if $i < j$, then

$$2\left[\left(f_{ij}' M f_{ij}\right)^2 + \left(g_{ij}' M g_{ij}\right)^2\right] = \left(M_{ii} + M_{jj}\right)^2 + \left(M_{ij} + M_{ji}\right)^2 \geq \left(M_{ij} + M_{ji}\right)^2,$$

where $f_{ij} = (e_i + e_j)/\sqrt{2}$ and $g_{ij} = (e_i - e_j)/\sqrt{2}$. As a consequence, the desired result can be obtained by setting $C = 2C_d = d(d+1)$, $J = d^2$, and letting,

$$\{l_i : 1 \leq i \leq J\} = \{e_i : 1 \leq i \leq d\} \cup \{f_{ij} : 1 \leq i < j \leq d\} \cup \{g_{ij} : 1 \leq i < j \leq d\}.$$

5.2 Central Limit Theorem

Lemma 5.2: Suppose the assumptions of Theorem 2 in the main paper hold, then

$$\sup_{\lambda \in \Lambda} \sup_{t \in \mathbb{R}} \left| \mathbb{P}^* \left[\frac{\lambda' \left(\hat{\theta}_m^*(h) - \mathbb{E}^* \left[\hat{\theta}_m^*(h) \right] \right)}{\sqrt{\lambda' \mathbb{V}^* \left[\hat{\theta}_m^*(h) \right] \lambda}} \leq t \right] - \Phi(t) \right| \longrightarrow_p 0,$$

where

$$\mathbb{V}^* \left[\hat{\theta}_m^*(h)\right] = \frac{1}{m} \mathbb{V}^* \left[L^*(z_i^*; h)\right] + \binom{m}{2}^{-1} h_m^{-(d+2)} \mathbb{V}^* \left[W^*\left(z_i^*, z_j^*; h\right)\right].$$

Proof of Lemma 5.2: The proof follows by applying the theorem of Heyde and Brown (1970). Conditional on Z_n, let $\mathcal{F}_m^* = \sigma(z_1^*, z_2^*, \ldots, z_m^*)$ and note that $L_m^*(z_i^*) \in \mathcal{F}_i^*$ with $\mathbb{E}^*[L_m^*(z_i^*)] = 0$, and $W_m^*\left(z_i^*, z_j^*\right) \in \mathcal{F}_i^*$

$(j < i)$ with $\mathbb{E}^*\left[W_m^*\left(z_i^*, z_j^*\right)\middle| z_i^*\right] = 0$. Define,

$$\begin{aligned} X_m^*(\lambda; h) &= \sum_{i=1}^m Y_i^*(\lambda; h), \\ Y_i^*(\lambda; h) &= m^{-1}\left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*(h)\right]\lambda\right)^{-1/2}\lambda' L^*\left(z_i^*; h\right) + \sum_{j=1}^{i-1}\binom{m}{2}^{-1}\left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*(h)\right]\lambda\right)^{-1/2}\lambda' W^*\left(z_i^*, z_j^*; h\right), \end{aligned}$$

and note that $(X_m^*, \mathcal{F}_m^*)$ is a martingale-difference sequence with

$$\left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*(h)\right]\lambda\right)^{-1/2}\left(\lambda'\left(\hat{\theta}_m^*(h) - \mathbb{E}\left[\hat{\theta}_m^*(h)\right]\right)\right) = X_m^*(\lambda; h).$$

Before proceeding on note that by standard matrix properties $\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*(h)\right]\lambda \geq \underline{\zeta}^*$ where $\underline{\zeta}^*$ is the minimum eigenvalue of the matrix $\mathbb{V}^*\left[\hat{\theta}_m^*(h)\right]$. By our assumptions, there exists a constant C such that $\underline{\zeta}^* \geq C\max\left(m^{-1}, m^{-2}h_m^{-(d+2)}\right)$. Then, by it follows that for sufficiently large n, there exists a constant C (which is not a function of λ) such that,

$$\left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*(h)\right]\lambda\right)^{-2} \leq C\min\left(m^2, m^4 h^{2(d+2)}\right),$$

with arbitrarily high probability.

Now, by Heyde and Brown (1970),

$$\begin{aligned} &\sup_{\lambda\in\Lambda}\sup_{t\in\mathbb{R}}\left|\mathbb{P}^*\left[\frac{\lambda'\left(\hat{\theta}_m^*(h) - \mathbb{E}^*\left[\hat{\theta}_m^*\right]\right)}{\sqrt{\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*(h)\right]\lambda}} \leq t\right] - \Phi(t)\right| \\ \leq\ & C\sup_{\lambda\in\Lambda}\left\{\sum_{i=1}^m \mathbb{E}^*\left[\left(Y_i^*(\lambda; h)\right)^4\right] + \mathbb{E}^*\left[\left(\sum_{i=1}^m \mathbb{E}^*\left[\left(Y_i^*(\lambda; h)\right)^2\middle|\mathcal{F}_{i-1}^*\right] - 1\right)^2\right]\right\}^{1/5}. \end{aligned}$$

Thus, it suffices to show that,

$$\sup_{\lambda\in\Lambda}\sum_{i=1}^m \mathbb{E}^*\left[\left(Y_i^*(\lambda; h)\right)^4\right] \longrightarrow_p 0 \tag{1}$$

and

$$\sup_{\lambda\in\Lambda}\mathbb{E}^*\left[\left(\sum_{i=1}^m \mathbb{E}^*\left[\left(Y_i^*(\lambda; h)\right)^2\middle|\mathcal{F}_{i-1}^*\right] - 1\right)^2\right] \longrightarrow_p 0. \tag{2}$$

First consider equation (1). Note that,

$$\sum_{i=1}^m \mathbb{E}^*\left[\left(Y_i^*(\lambda; h)\right)^4\right] = \left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*(h)\right]\lambda\right)^{-2}\sum_{i=1}^m \mathbb{E}^*\left[\left(\lambda' M_{1,n,i}\lambda\right)^2\right],$$

where $M_{1,n,i} = \tilde{M}_{1,n,i}\tilde{M}'_{1,n,i}$ and

$$\tilde{M}_{1,n,i} = m^{-1}L^*\left(z_i^*;h\right) + \binom{m}{2}^{-1}\sum_{j=1}^{i-1} W^*\left(z_i^*, z_j^*;h\right).$$

Thus by the above discussion we need only show that,

$$O_p\left(\min\left(m^2, m^4h^{2(d+2)}\right)\right)\sup_{\lambda\in\Lambda}\sum_{i=1}^{m}\mathbb{E}^*\left[\left(\lambda' M_{1,n,i}\lambda\right)^2\right] = o_p\left(1\right).$$

Furthermore, by Lemma 4.4.1 and Markov's inequality it suffices to show,

$$O_p\left(\min\left(m^2, m^4h^{2(d+2)}\right)\right)\sum_{i=1}^{m}\mathbb{E}\left[\left(\lambda' M_{1,n,i}\lambda\right)^2\right] = o_p\left(1\right), \tag{3}$$

Now consider equation (2). Note that,

$$\begin{aligned}
&\sum_{i=1}^{m}\mathbb{E}^*\left[\left(Y_i^*\left(\lambda;h\right)\right)^2\Big|\,\mathcal{F}_{i-1}^*\right] \\
=\ & \left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*\left(h\right)\right]\lambda\right)^{-1}\mathbb{E}^*\left[\left(\left[m^{-1}\sum_{i=1}^{m}\lambda' L^*\left(z_i^*;h\right)\right]\right)^2\right] \\
& + \left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*\left(h\right)\right]\lambda\right)^{-1}\binom{m}{2}^{-2}\sum_{i=1}^{m}\mathbb{E}^*\left[\left(\sum_{j=1}^{i-1}\lambda' W^*\left(z_i^*, z_j^*;h\right)\right)^2\Bigg|\,\mathcal{F}_{i-1}^*\right] \\
& + 2m^{-1}\binom{m}{2}^{-1}\left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*\left(h\right)\right]\lambda\right)^{-1}\sum_{i=1}^{m}\sum_{j=1}^{i-1}\mathbb{E}^*\left[\left(\lambda' L^*\left(z_i^*;h\right)\right)\left(\lambda' W^*\left(z_i^*, z_j^*;h\right)\right)\big|\,\mathcal{F}_{i-1}^*\right] \\
=\ & 1 + \left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*\left(h\right)\right]\lambda\right)^{-1}\binom{m}{2}^{-2}\sum_{i=1}^{m}\mathbb{E}^*\left[\left(\sum_{j=1}^{i-1}\lambda' W^*\left(z_i^*, z_j^*;h\right)\right)^2\Bigg|\,\mathcal{F}_{i-1}^*\right] \\
& - \left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*\left(h\right)\right]\lambda\right)^{-1}\binom{m}{2}^{-2}\sum_{i=1}^{m}\mathbb{E}^*\left[\left(\sum_{j=1}^{i-1}\lambda' W^*\left(z_i^*, z_j^*;h\right)\right)^2\right] \\
& + 2m^{-1}\binom{m}{2}^{-1}\left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*\left(h\right)\right]\lambda\right)^{-1}\sum_{i=1}^{m}\sum_{j=1}^{i-1}\mathbb{E}^*\left[\left(\lambda' L^*\left(z_i^*;h\right)\right)\left(\lambda' W^*\left(z_i^*, z_j^*;h\right)\right)\big|\,\mathcal{F}_{i-1}^*\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}^*\left[\left(\sum_{i=1}^{m}\mathbb{E}^*\left[\left(Y_i^*\left(\lambda;h\right)\right)^2\middle|\mathcal{F}_{i-1}^*\right]-1\right)^2\right] \\
=\;&\left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*\left(h\right)\right]\lambda\right)^{-2} \\
&\mathbb{E}^*\left[\left(\binom{m}{2}^{-2}\sum_{i=1}^{m}\mathbb{E}^*\left[\left(\sum_{j=1}^{i-1}\lambda'W^*\left(z_i^*,z_j^*;h\right)\right)^2\middle|\mathcal{F}_{i-1}^*\right]-\mathbb{E}^*\left[\left(\sum_{j=1}^{i-1}\lambda'W^*\left(z_i^*,z_j^*;h\right)\right)^2\right]\right.\right. \\
&\left.\left.+\;2m^{-1}\binom{m}{2}^{-1}\sum_{i=1}^{m}\sum_{j=1}^{i-1}\mathbb{E}^*\left[\left(\lambda'L^*\left(z_i^*;h\right)\right)\left(\lambda'W^*\left(z_i^*,z_j^*;h\right)\right)\middle|\mathcal{F}_{i-1}^*\right]\right)^2\right].
\end{aligned}$$

Consequently,

$$\mathbb{E}^*\left[\left(\sum_{i=1}^{m}\mathbb{E}^*\left[\left(Y_i^*\left(\lambda;h\right)\right)^2\middle|\mathcal{F}_{i-1}^*\right]-1\right)^2\right]=\left(\lambda'\mathbb{V}^*\left[\hat{\theta}_m^*\left(h\right)\right]\lambda\right)^{-2}\mathbb{E}^*\left[\left(\lambda'M_{2,n}\lambda\right)^2\right],$$

where

$$\begin{aligned}
M_{2,n} \;=\;& \binom{m}{2}^{-2}\sum_{i=1}^{m}\sum_{j=1}^{i-1}\sum_{k=1}^{i-1}\left(\mathbb{E}^*\left[W^*\left(z_i^*,z_j^*;h\right)W^*\left(z_i^*,z_k^*;h\right)'\middle|\mathcal{F}_{i-1}^*\right]-\mathbb{E}^*\left[W^*\left(z_i^*,z_j^*;h\right)W^*\left(z_i^*,z_k^*;h\right)'\right]\right) \\
&+2m^{-1}\binom{m}{2}^{-1}\sum_{i=1}^{m}\sum_{j=1}^{i-1}\mathbb{E}^*\left[L^*\left(z_i^*;h\right)W^*\left(z_i^*,z_j^*;h\right)'\middle|\mathcal{F}_{i-1}^*\right].
\end{aligned}$$

Thus by the above discussion we need only show that,

$$O_p\left(\min\left(m^2,m^4h^{2(d+2)}\right)\right)\sup_{\lambda\in\Lambda}\mathbb{E}^*\left[\left(\lambda'M_{2,n}\lambda\right)^2\right]=o_p\left(1\right).$$

Furthermore, by Lemma 4.4.1 and Markov's inequality it suffices to show,

$$O_p\left(\min\left(m^2,m^4h^{2(d+2)}\right)\right)\mathbb{E}\left[\left(\lambda'M_{2,n}\lambda\right)^2\right]=o_p\left(1\right) \tag{4}$$

Condition (3) holds because, by basic inequalities,

$$\sum_{i=1}^{m}\mathbb{E}\left[\left(\lambda'M_{1,n,i}\lambda\right)^2\right]\leq Cm^{-4}\sum_{i=1}^{m}\mathbb{E}\left[\left(\lambda'L^*\left(z_i^*;h\right)\right)^4\right]+Cm^{-8}\sum_{i=1}^{m}\mathbb{E}\left[\left(\sum_{j=1}^{i-1}\lambda'W^*(z_i^*,z_j^*;h)\right)^4\right].$$

After the appropriate normalization, the first term becomes,

$$\begin{aligned}
& m^{-4}\min\left(m^2, m^4 h^{2(d+2)}\right)\sum_{i=1}^{m}\mathbb{E}\left[\left(L^*\left(z_i^*;h\right)\right)^4\right] \\
= \quad & \min\left(m^{-1}, mh^{2(d+2)}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*[\lambda' U_{h,ij}^* | z_i^*]\right)^4\right]\right] \\
\leq \quad & \min\left(m^{-1}, mh^{2(d+2)}\right)\mathbb{E}\left[\frac{C}{n^3}\tilde{T}_{1,n}^{(4)}\left(\lambda;h\right)+\frac{C}{n^2}\tilde{T}_{2,n}^{(2)}\left(\lambda;h\right)+\frac{C}{n}\tilde{T}_{4,n}\left(\lambda;h\right)+C\tilde{T}_{7,n}\left(\lambda;h\right)\right] \\
= \quad & C\min\left(m^{-1}, mh^{2(d+2)}\right)\left(n^{-3}h^{-3d-4}+n^{-2}h^{-2d-4}+n^{-1}h^{-d-2}+1\right) \\
\to \quad & 0, \qquad \text{if } m^2h^d \to \infty,
\end{aligned}$$

by Lemma 4.2.3, Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.4 and Lemma 3.2.7. After the appropriate normalization, the second term becomes,

$$\begin{aligned}
& m^{-8}\min\left(m^2, m^4 h^{2(d+2)}\right)\sum_{i=1}^{m}\mathbb{E}\left[\left(\sum_{j=1}^{i-1}\lambda' W^*(z_i^*, z_j^*;h)\right)^4\right] \\
\leq \quad & C\min\left(m^{-4}, m^{-2}h^{2(d+2)}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)^4\right]\right] \\
& +C\min\left(m^{-3}, m^{-1}h^{2(d+2)}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)^2\left(\lambda' W^*(z_i^*, z_k^*;h)\right)^2\right]\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\min\left(m^{-4}, m^{-2}h^{2(d+2)}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)^4\right]\right] \quad \leq \quad & C\min\left(m^{-4}, m^{-2}h^{2(d+2)}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' U_{h,ij}^*\right)^4\right]\right] \\
= \quad & C\min\left(m^{-4}, m^{-2}h^{2(d+2)}\right)\mathbb{E}\left[\tilde{T}_{1,n}^{(4)}\left(\lambda;h\right)\right] \\
= \quad & C\min\left(m^{-4}, m^{-2}h^{2(d+2)}\right)h^{-3d-4} \\
\to \quad & 0, \qquad \text{if } m^2h^d \to \infty,
\end{aligned}$$

by Lemma 4.2.1 and Lemma 3.2.1 and

$$\begin{aligned}
& \min\left(m^{-3}, m^{-1}h^{2(d+2)}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)^2\left(\lambda' W^*(z_i^*, z_k^*;h)\right)^2\right]\right] \\
\leq \quad & C\min\left(m^{-3}, m^{-1}h^{2(d+2)}\right)\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)^2\middle| z_i^*\right]\right)^2\right] \\
\leq \quad & C\min\left(m^{-3}, m^{-1}h^{2(d+2)}\right)\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\lambda' U_{h,ij}^*\right)^2\middle| z_i^*\right]\right)^2\right] \\
\leq \quad & C\min\left(m^{-3}, m^{-1}h^{2(d+2)}\right)\mathbb{E}\left[\frac{1}{n}\tilde{T}_{1,n}^{(4)}\left(\lambda;h\right)+\tilde{T}_{2,n}^{(2)}\left(\lambda;h\right)\right] \\
= \quad & C\min\left(m^{-3}, m^{-1}h^{2(d+2)}\right)\left(n^{-1}h^{-3d-4}+h^{-2d-4}\right) \\
\to \quad & 0, \qquad \text{if } m^2h^d \to \infty.
\end{aligned}$$

by Lemma 4.2.2, Lemma 3.2.1 and Lemma 3.2.2. Thus, equation (3) holds.

Now consider equation (4). Note that equation (4) is bounded above by $2R^*_{1,m}(\lambda;h) + 2R^*_{2,m}(\lambda;h)$ where,

$$R^*_{1,m}(\lambda;h) = \binom{m}{2}^{-4} \mathbb{E}\left[\left(\sum_{i=1}^{m}\left\{\mathbb{E}^*\left[\left(\sum_{j=1}^{i-1}\lambda' W^*(z_i^*,z_j^*;h)\right)^2\middle|\mathcal{F}^*_{i-1}\right] - \mathbb{E}^*\left[\left(\sum_{j=1}^{i-1}\lambda' W^*_{m,n}(z_i^*,z_j^*;h)\right)^2\right]\right\}\right)^2\right],$$

and

$$R^*_{2,m}(\lambda;h) = m^{-2}\binom{m}{2}^{-2}\mathbb{E}\left[\left(\sum_{i=1}^{m}\sum_{j=1}^{i-1}\mathbb{E}^*\left[\left(\lambda' L^*(z_i^*;h)\right)\left(\lambda' W^*(z_i^*,z_j^*;h)\right)|z_j^*\right]\right)^2\right].$$

For the first remainder, $R^*_{1,m}(\lambda;h)$, note that

$$\left(\sum_{j=1}^{i-1}\lambda' W^*(z_i^*,z_j^*;h)\right)^2 = \sum_{j=1}^{i-1}\left(\lambda' W^*(z_i^*,z_j^*;h)\right)^2 + 2\sum_{j=1}^{i-1}\sum_{k=1}^{j-1}\left(\lambda' W^*(z_i^*,z_j^*;h)\right)\left(\lambda' W^*(z_i^*,z_k^*;h)\right),$$

and therefore

$$R^*_{1,m}(\lambda;h) \leq \binom{m}{2}^{-4} R^*_{11,m}(\lambda;h) + 2\binom{m}{2}^{-4} R^*_{12,m}(\lambda;h),$$

where

$$R^*_{11,m}(\lambda;h) = \mathbb{E}\left[\left(\sum_{i=1}^{m}\sum_{j=1}^{i-1}\left\{\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*,z_j^*;h)\right)^2\middle|\mathcal{F}^*_{i-1}\right] - \mathbb{E}^*\left[\left(\lambda' W^*(z_i^*,z_j^*;h)\right)^2\right]\right\}\right)^2\right],$$

and

$$R^*_{12,m}(\lambda;h) = \mathbb{E}\left[\left(\sum_{i=1}^{m}\sum_{j=1}^{i-1}\sum_{k=1}^{j-1}\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*,z_j^*;h)\right)\left(\lambda' W^*(z_i^*,z_k^*;h)\right)\middle|\mathcal{F}^*_{i-1}\right]\right)^2\right].$$

Next, note that $\sum_{i=1}^{n}\sum_{j=1}^{i-1} a_{ij} = \sum_{j=1}^{n-1}\sum_{i=j+1}^{n} a_{ij}$, and hence

$$\begin{aligned} R^*_{11,m}(\lambda;h) &= \mathbb{E}\left[\left(\sum_{i=1}^{m}\sum_{j=1}^{i-1}\left\{\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*,z_j^*;h)\right)^2\middle|z_j^*\right] - \mathbb{E}^*\left[\left(\lambda' W^*(z_i^*,z_j^*;h)\right)^2\right]\right\}\right)^2\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{m}(m-i)\left\{\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*,z_j^*;h)\right)^2\middle|z_i^*\right] - \mathbb{E}^*\left[\left(\lambda' W^*(z_i^*,z_j^*;h)\right)^2\right]\right\}\right)^2\right] \\ &\leq Cm^3\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*,z_j^*;h)\right)^2\middle|z_i^*\right]\right)^2\right], \end{aligned}$$

which gives, after the appropriate normalization,

$$\begin{aligned}\min\left(m^2, m^4 h^{2(d+2)}\right)\binom{m}{2}^{-4} R^*_{11,m}(\lambda;h) &\leq C\min\left(m^{-3}, m^{-1}h^{2(d+2)}\right)\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)^2\Big| z_i^*\right]\right)^2\right] \\ &\leq C\min\left(m^{-3}, m^{-1}h^{2d+4}\right)\mathbb{E}\left[\frac{1}{n}\tilde{T}^{(4)}_{1,n}(\lambda;h) + \tilde{T}^{(2)}_{2,n}(\lambda;h)\right] \\ &\to 0, \qquad \text{if } m^2h^d \to \infty,\end{aligned}$$

by Lemma 4.2.2, Lemma 3.2.1 and Lemma 3.2.2. Next, note that

$$\begin{aligned}& R^*_{12,m}(\lambda;h) \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{m}\sum_{j=1}^{i-1}\sum_{k=1}^{j-1}\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\right)^2\right] \\ &= \mathbb{E}\left[\left(\sum_{j=1}^{m-1}\sum_{k=1}^{j-1}(m-j)\,\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\right)^2\right]\end{aligned}$$

$$= \sum_{j=1}^{m-1}\sum_{k=1}^{j-1}(m-j)^2\,\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\right)^2\right]$$

$$+ \sum_{j=1}^{m-1}\sum_{k=1}^{j-1}\sum_{h=1,h\neq k}^{j-1}(m-j)^2 \times$$
$$\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_h^*;h)\right)\big| z_j^*, z_h^*\right]\right]$$

$$+ \sum_{j=1}^{m-1}\sum_{k=1}^{j-1}\sum_{l=1,l\neq j}^{m-1}\sum_{h=1}^{l-1}(m-j)(m-l) \times$$
$$\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_l^*;h)\right)\left(\lambda' W^*(z_i^*, z_h^*;h)\right)\big| z_l^*, z_h^*\right]\right]$$

$$\leq Cm^4\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\right)^2\right],$$

because, for $h \neq k$,

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_h^*;h)\right)\big| z_j^*, z_h^*\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}^*\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_h^*;h)\right)\big| z_j^*, z_h^*\right]\big| z_j^*\right]\right] \\ &= \mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*\right]\right)^2\right] = 0,\end{aligned}$$

and, for $l \neq j$,

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_l^*;h)\right)\left(\lambda' W^*(z_i^*, z_h^*;h)\right)\big| z_l^*, z_h^*\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_j^*, z_k^*\right]\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_l^*;h)\right)\left(\lambda' W^*(z_i^*, z_h^*;h)\right)\big| z_l^*, z_h^*\right]\big| z_k^*, z_h^*\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\big| z_k^*\right]\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_l^*;h)\right)\left(\lambda' W^*(z_i^*, z_h^*;h)\right)\big| z_h^*\right]\right] = 0.\end{aligned}$$

Therefore, after the appropriate normalization,

$$
\begin{aligned}
&\min\left(m^2, m^4 h^{2(d+2)}\right)\binom{m}{2}^{-4} R^*_{12,m}(\lambda;h) \\
\leq\ & C\min\left(m^{-2}, h^{2(d+2)}\right)\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\lambda' W^*(z_i^*, z_j^*;h)\right)\left(\lambda' W^*(z_i^*, z_k^*;h)\right)\middle| z_j^*, z_k^*\right]\right)^2\right] \\
\leq\ & C\min\left(m^{-2}, h^{2d+4}\right)\mathbb{E}\left[\frac{1}{n^2}\tilde{T}^{(4)}_{1,n}(\lambda;h) + \frac{C}{n}\tilde{T}^{(2)}_{2,n}(\lambda;h) + C\tilde{T}_{6,n}(\lambda;h)\right] \\
=\ & C\min\left(m^{-2}, h^{2d+4}\right)\left(n^{-2}h^{-3d-4} + n^{-1}h^{-2d-4} + h^{-d-4}\right) \\
\to\ & 0, \qquad \text{if } m^2 h^d \to \infty,
\end{aligned}
$$

by Lemma 4.2.7, Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.6.

Finally, for the second remainder, $R^*_{2,m}(\lambda;h)$ with the appropriate normalization, is

$$
\begin{aligned}
&\min\left(m^2, m^4 h^{2(d+2)}\right) R^*_{2,m}(\lambda;h) \\
&= m^{-2}\binom{m}{2}^{-2}\min\left(m^2, m^4 h^{2(d+2)}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\sum_{i=1}^{m}\sum_{j=1}^{i-1}\mathbb{E}^*\left[\left(\lambda' L^*(z_i^*;h)\right)\left(\lambda' W^*(z_i^*, z_j^*;h)\right)|z_j^*\right]\right)^2\right]\right] \\
&\leq C\min\left(m^{-1}, mh^{2d+4}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\lambda' L^*(z_i^*;h)\right)\left(\lambda' W^*(z_i^*, z_j^*;h)\right)|z_j^*\right]\right)^2\right]\right] \\
&\leq C\min\left(m^{-1}, mh^{2d+4}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\mathbb{E}^*[\lambda' U^*_{h,ij}|z_i^*] - \lambda'\theta^*(h)\right)\left(\lambda' U^*_{h,ij} - \mathbb{E}^*[\lambda' U^*_{h,ij}|z_i^*]\right)\middle| z_j^*\right]\right)^2\right]\right] \\
&\leq C\min\left(m^{-1}, mh^{2d+4}\right)\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\mathbb{E}^*[\lambda' U^*_{h,ij}|z_i^*] - \lambda'\theta^*(h)\right)\left(\lambda' U^*_{h,ij}\right)\middle| z_j^*\right]\right)^2\right]\right] \\
&\leq C\min\left(m^{-1}, mh^{2d+4}\right)\left(n^{-3}h^{-3d-4} + n^{-2}h^{-2d-4} + n^{-1}h^{-d-2} + n^{-1}h^{-d-3} + n^{-1}h^{-d-4} + 1\right) \\
&\qquad + C\min\left(m^{-1}, mh^{2d+4}\right)\left(n^{-2}h^{-2d-4} + n^{-4}h^{-3d-4} + 1 + n^{-2}h^{-d-4} + n^{-3}h^{-2d-4}\right) \\
&\to 0, \qquad \text{if } m^2 h^d \to \infty.
\end{aligned}
$$

because

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\left(\mathbb{E}^*[\lambda' U^*_{h,ij}|z_i^*] - \lambda'\theta^*(h)\right)\left(\lambda' U^*_{h,ij}\right)\middle| z_j^*\right]\right)^2\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\mathbb{E}^*[\lambda' U^*_{h,ij}|z_i^*]\left(\lambda' U^*_{h,ij}\right)\middle| z_j^*\right] - \left(\lambda'\theta^*(h)\right)\mathbb{E}^*\left[\lambda' U^*_{h,ij}\middle| z_j^*\right]\right)^2\right]\right] \\
&\leq C\mathbb{E}\left[\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\mathbb{E}^*[\lambda' U^*_{h,ij}|z_i^*]\left(\lambda' U^*_{h,ij}\right)\middle| z_j^*\right]\right)^2\right]\right] + C\mathbb{E}\left[\left(\mathbb{E}^*\left[\left(\mathbb{E}^*\left[\lambda' U^*_{h,ij}\middle| z_j^*\right]\right)^2\right]\right)^2\right] \\
&\leq \frac{C}{n^3}\mathbb{E}\left[\tilde{T}^{(4)}_{1,n}(\lambda;h)\right] + \frac{C}{n^2}\mathbb{E}\left[\tilde{T}^{(2)}_{2,n}(\lambda;h)\right] + \frac{C}{n^2}\mathbb{E}\left[\tilde{T}_{9,n}(\lambda;h)\right] + \frac{C}{n}\mathbb{E}\left[\tilde{T}_{4,n}(\lambda;h)\right] \\
&+ \frac{C}{n}\mathbb{E}\left[\tilde{T}_{5,n}(\lambda;h)\right] + \frac{C}{n}\mathbb{E}\left[\tilde{T}_{6,n}(\lambda;h)\right] + C\mathbb{E}\left[\tilde{T}_{8,n}(\lambda;h)\right] + \frac{C}{n^2}\mathbb{E}\left[\left(\tilde{T}^{(2)}_{1,n}(\lambda;h)\right)^2\right] + C\mathbb{E}\left[\left(\tilde{T}^{(1)}_{2,n}(\lambda;h)\right)^2\right],
\end{aligned}
$$

where the bounds follow by Lemma 4.2.6 and Lemma 4.2.2 and the rates are obtained from Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.9, Lemma 3.2.4, Lemma 3.2.5, Lemma 3.2.6 and Lemma 3.2.8, and

$$
\mathbb{E}\left[\left(\tilde{T}^{(2)}_{1,n}(\lambda;h)\right)^2\right] = O\left(h^{-2d-4} + n^{-2}h^{-3d-4}\right),
$$

$$\mathbb{E}\left[\left(\tilde{T}_{2,n}^{(1)}\left(\lambda;h\right)\right)^{2}\right]=O\left(1+n^{-2}h^{-d-4}+n^{-3}h^{-2d-4}\right),$$

which follow by Lemma 3.1.4 and Lemma 3.1.5.

6 References

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