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Supplemental Appendix
"Characteristic-Sorted Portfolios: Estimation and Inference"

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A Optimal Choice of J_t

A.1 Theoretical Quantities

Here we provide explicit formulas for the main terms of the MSE expansion given in Theorem 3. First let us define $q_{jt} = \mathbb{P}\left(\mathbf{z} \in P_{jt} \middle| \mathcal{F}_t\right)$. Then, we have that:

$$\mathcal{B}_t\left(\mathbf{z}\right) = JT^{-1}n_t^{-1}\mu'(\mathbf{z})\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^d}\mathbb{1}_{jt}(\mathbf{z})q_{jt}^{-1}\mathbb{1}_{jt}(\mathbf{z}_{it})(\mathbf{z}_{it}-\mathbf{z})$$

where $\mu'(\mathbf{z}_0) = \left.\partial\mu(\mathbf{z})/\partial\mathbf{z}'\right|_{\mathbf{z}=\mathbf{z}_0}$ and

$$\mathcal{V}_t^{(1)}\left(\mathbf{z}\right) = nJ^{-d}T^{-1}n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^d}\mathbb{1}_{jt}(\mathbf{z})q_{jt}^{-2}\mathbb{E}\left[\mathbb{1}_{jt}\left(\mathbf{z}_{it}\right)\sigma_{it}^2\middle|\mathcal{F}_t\right],$$
$$\mathcal{V}_t^{(2)}\left(\mathbf{z}\right) = n^2J^{-2d}T^{-1}n_t^{-3}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^d}\mathbb{1}_{jt}\left(\mathbf{z}\right)q_{jt}^{-3}\mathbb{E}\left[\mathbb{1}_{jt}\left(\mathbf{z}_{it}\right)\sigma_{it}^2\middle|\mathcal{F}_t\right].$$

Finally, we have $\mathcal{C} = \sum_{t=1}^{T}\mathcal{C}_t\left(\mathbf{z}_L\right) + \sum_{t=1}^{T}\mathcal{C}_t\left(\mathbf{z}_H\right)$ where,

$$\begin{aligned}\mathcal{C}_t\left(\mathbf{z}\right) &= n^{3/2}J^{-3d/2}T^{-1/2}n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^d}\mathbb{1}_{jt}\left(\mathbf{z}\right)q_{jt}^{-2}\left\{\mathbb{1}_{jt}\left(\mathbf{z}_{it}\right)\sigma_{it}^2 - \mathbb{E}\left[\mathbb{1}_{jt}\left(\mathbf{z}_{it}\right)\sigma_{it}^2\middle|\mathcal{F}_t\right]\right\}\\ &\quad - 2n^{3/2}J^{-3d/2}T^{-1/2}\sum_{t=1}^{T}n_t^{-3}\sum_{i_1\neq i_2}^{n_t}\sum_{j=1}^{J_t^d}\mathbb{1}_{jt}\left(\mathbf{z}\right)q_{jt}^{-3}\left(\mathbb{1}_{jt}\left(\mathbf{z}_{i_2t}\right)-q_{jt}\right)\mathbb{1}_{jt}\left(\mathbf{z}_{i_1t}\right)\sigma_{i_1t}^2.\end{aligned}$$

A.2 Empirical Implementation

As we discussed in Section 5 we base our choice of the optimal number of portfolios in our empirical applications based on equation (11). To do so let $t_{\max} = \arg\max_{1\leq t\leq T} n_t$, $n = n_{t_{\max}}$ and $J = J_{t_{\max}}$. For all other time periods we scale J_t as $J_t = J\left(n_t/n\right)^{\frac{1}{d+1}}$ (see discussion in Section 4). We then choose a grid of values for J as $J = \left(\left(n_{t_{\min}}/n\right)^{\frac{1}{d+1}}, \ldots, J_{\max}\right)$ where $t_{\min} = \arg\min_{1\leq t\leq T} n_t$. In our empirical applications we set $J_{\max} = 400$.

To estimate the MSE in practice we have the following estimator,

$$\begin{aligned}&\widehat{\mathrm{MSE}}\left(\hat{\mu}(\mathbf{z}_H) - \hat{\mu}(\mathbf{z}_L); J_1, \ldots, J_T\right)\\ &= \left(\hat{\mu}'\left(\mathbf{z}_H\right)\cdot T^{-1}\sum_{t=1}^{T}\sum_{j=1}^{J_t^d}\sum_{i=1}^{n_t}\omega_{it}\hat{\mathbf{1}}_{jt}\hat{\mathbb{1}}_{jt}(\mathbf{z}_H)\hat{\mathbb{1}}_{jt}(\mathbf{z}_{it})\left(\mathbf{z}_{it}-\mathbf{z}_H\right)\right.\\ &\qquad\left. -\hat{\mu}'\left(\mathbf{z}_L\right)\cdot T^{-1}\sum_{t=1}^{T}\sum_{j=1}^{J_t^d}\sum_{i=1}^{n_t}\omega_{it}\hat{\mathbf{1}}_{jt}\hat{\mathbb{1}}_{jt}(\mathbf{z}_L)\hat{\mathbb{1}}_{jt}(\mathbf{z}_{it})\left(\mathbf{z}_{it}-\mathbf{z}_L\right)\right)^2\\ &\qquad + T^{-2}\sum_{t=1}^{T}\left(\hat{m}_t\left(\mathbf{z}_H\right)-\hat{m}_t\left(\mathbf{z}_L\right)-\left(\hat{m}\left(\mathbf{z}_H\right)-\hat{m}\left(\mathbf{z}_L\right)\right)\right)^2\end{aligned}\tag{A.1}$$

where

$$\hat{m}_t\left(\mathbf{z}\right) = \sum_{j=1}^{J_t^d}N_{jt}^{-1/2}\sum_{i=1}^{n_t}\omega_{it}\hat{\mathbf{1}}_{jt}\hat{\mathbb{1}}_{jt}(\mathbf{z})\hat{\mathbb{1}}_{jt}(\mathbf{z}_{it})(R_{it}-\mathbf{x}_{it}'\hat{\boldsymbol{\beta}}_t), \qquad \hat{m}\left(\mathbf{z}\right) = T^{-1}\sum_{t=1}^{T}\hat{m}_t\left(\mathbf{z}\right).$$

Here ω_{it} is the weight applied to the returns in each portfolio which satisfies $\sum_{i=1}^{n_t}\hat{\mathbb{1}}_{jt}(\mathbf{z}_{it})\omega_{it} = 1$ for each

$j = 1, \ldots, J_t^d$ and at each time t. As is common, we use lagged market equity to weight the returns in each portfolio in our empirical applications. The plug-in estimate of $\hat{\mathcal{V}}^{(2)} \frac{J^{2d}}{n^2 T}$ implicit in the above expression utilizes the logic of the Fama-MacBeth variance estimator applied to the higher-order variance term. As a plug-in estimator of $\mu'(\mathbf{z})$ we use the time-series average of the estimated slope coefficient from a local regression using the 40 closest points to $\mathbf{z}$ (ties included) at each point in time.

Remark 1. As discussed in Remark 9 of the main text, when we are interested in point estimation, the optimal choice is $J_t^{\star\star}$ rather than $J_t^\star$. In analogy with equation (A.1) we can utilize the following estimator,

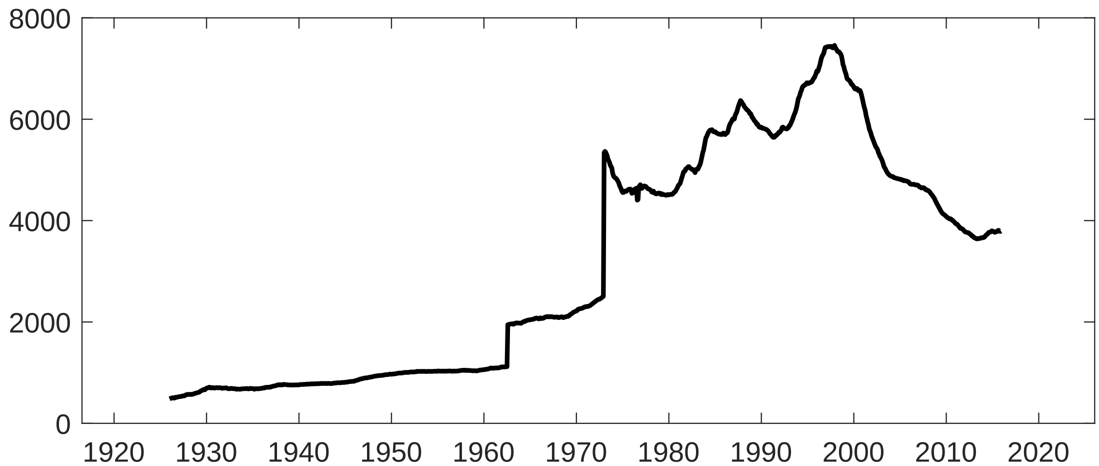
$$\begin{aligned}
&\widehat{\mathbb{MSE}}^{\star\star}\left(\hat{\mu}(\mathbf{z}_H) - \hat{\mu}(\mathbf{z}_L); J_1, \ldots, J_T\right) \\
&= \Bigg(\hat{\mu}'(\mathbf{z}_H) \cdot T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{J_t^d} \sum_{i=1}^{n_t} \omega_{it} \hat{\mathbf{1}}_{jt} \hat{\mathbb{1}}_{jt}(\mathbf{z}_H) \hat{\mathbb{1}}_{jt}(\mathbf{z}_{it}) \left(\mathbf{z}_{it} - \mathbf{z}_H\right) \\
&\qquad - \hat{\mu}'(\mathbf{z}_L) \cdot T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{J_t^d} \sum_{i=1}^{n_t} \omega_{it} \hat{\mathbf{1}}_{jt} \hat{\mathbb{1}}_{jt}(\mathbf{z}_L) \hat{\mathbb{1}}_{jt}(\mathbf{z}_{it}) \left(\mathbf{z}_{it} - \mathbf{z}_L\right)\Bigg)^2 + \hat{V}_{\mathtt{FM}}(\mathbf{z}).
\end{aligned}$$

In this case, we would scale all other time periods as $J_t = J\left(n_t/n\right)^{\frac{1}{d+2}}$. We then choose a grid of values for J as $J = \left(\left(n_{t_{\min}}/n\right)^{\frac{1}{d+2}}, \ldots, J_{\max}\right)$. □

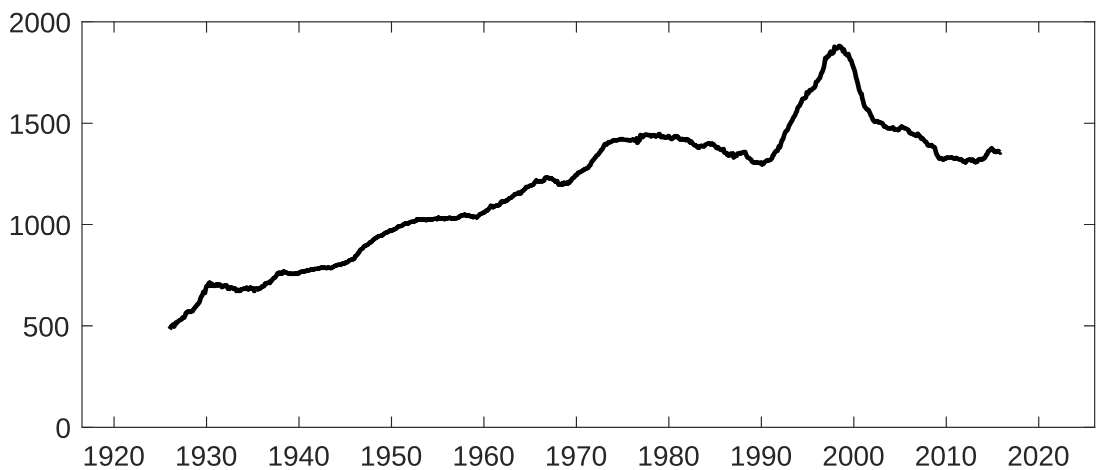
Figure A.1: **Cross-Sectional Sample Sizes**

The top chart shows the monthly cross-section sample sizes over time, n_t, for the primary data set from the Center for Research in Security Prices (CRSP). The bottom chart shows the cross-section sample sizes over time for those stocks listed on the New York Stock Exchange (NYSE).

All



NYSE Only



B Proofs

B.1 Notation

In this Supplementary Appendix we use a generalized notation relative to the manuscript. Note that N_{jt} satisfies $N_{jt} = n_t \hat{q}_{jt}$ where $\hat{q}_{jt}$ is defined below. The other mappings from the manuscript to remainder of this supplement are as follows:

- $\hat{\mathbf{1}}_{jt} \mapsto \mathbf{1}_{jt}$
- $\hat{\mathbb{1}}_{jt}(\mathbf{z}) \mapsto \widehat{\mathbb{I}}_{jt}(z)$
- $d \mapsto d_z$

We also abstain from bold symbols in the remainder of the supplement for simplicity of notation.

B.2 Model, Setup and Assumptions

Let $R_{it} \in \mathbb{R}$ be the return of asset i at time t with regressor of interest, $z_{it} \in \mathbb{R}^{d_z}$ and additional controls, $x_{it} \in \mathbb{R}^{d_x}$. The model is

$$R_{it} = \mu(z_{it}) + x_{it}'\beta_t + \varepsilon_{it}, \qquad i = 1, \ldots, n_t, \qquad t = 1, \ldots, T, \tag{$\star$}$$

where $\beta_t \in \mathbb{R}^{d_x}$ $\forall t$ and $\mu(\cdot)$ is a time-invariant function. We make the following assumptions:

Assumption 1. *Let the sigma fields $\mathcal{F}_t = \sigma(f_t)$ be generated by a sequence of unobserved (possibly dependent) random vectors $\{f_t : t = 0, 1, \ldots, T\}$. For $t = 1, \ldots, T$, the following conditions hold.*

1. Conditional on $\mathcal{F}_t$, $\{(R_{it}, z_{it}', x_{it}') : i = 1, 2, \ldots, n_t\}$ are iid satisfying Model ($\star$).
2. $\mathbb{E}[\varepsilon_{it} | z_{it}, x_{it}, \mathcal{F}_t] = 0$; uniformly in t, $\Omega_{\mathrm{uu},t} = \mathbb{E}[\mathbb{V}(x_{it} | z_{it}, \mathcal{F}_t) | \mathcal{F}_t]$ is bounded and its minimum eigenvalue is bounded away from zero; $\sigma_{it}^2 = \mathbb{E}\left[|\varepsilon_{it}|^2 \middle| z_{it}, x_{it}, \mathcal{F}_t \right]$ is bounded and bounded away from zero, and $\mathbb{E}\left[|\varepsilon_{it}|^{2+\phi} \middle| z_{it}, x_{it}, \mathcal{F}_t \right]$ is bounded for some $\phi > 0$; $\mathbb{E}[a' x_{it} | z_{it}, \mathcal{F}_t]$ is sub-Gaussian for all $a \in \mathbb{R}^{d_x}$.
3. Conditional on $\mathcal{F}_t$, z_{it} has time-invariant support, denoted $\mathcal{Z}$, and continuous Lebesgue density bounded away from zero.
4. $\mu(z)$ is twice continuously differentiable; $|\mathbb{E}[x_{it,\ell} | z_{it} = z, \mathcal{F}_t] - \mathbb{E}[x_{it,\ell} | z_{it} = z', \mathcal{F}_t]| \leq C \|z - z'\|$ for all all $z, z' \in \mathcal{Z}$ where $x_{it,\ell}$ is the ℓth element of x_{it} and the constant $C > 0$ is not a function of t or $\mathcal{F}_t$.

Assumption 2. *The cross-sectional sample sizes diverge proportionally for a sequence $n \to \infty$, $n_t = \kappa_t n$ with $\kappa_t \leq 1$ and uniformly bounded away from zero.*

Assumption 3. *The sequences n, T, and J obey: (a) $n^{-1} J^{d_z} \log(n) \log\left(J^{d_z} \vee T\right) \to 0$, (b) $\sqrt{nT} J^{-(d_z/2+1)} \to 0$, and, if $d_x \geq 1$, (c) $T/n \to 0$.*

Finally, let $\|A\| = \operatorname{tr}(A'A)$ for a matrix A. If A is square we denote the minimum eigenvalue by $\lambda_{\min}(A)$. Define for two sequences $a_{n,T} \asymp b_{n,T}$ if $\limsup_{n,T\to\infty} |a_{n,T}/b_{n,T}| < \infty$ and $\limsup_{n,T\to\infty} |b_{n,T}/a_{n,T}| < \infty$.

B.3 Estimation Approach

We approximate the unknown function $\mu(\cdot)$ at fixed time t by a partitioning estimator. At each point in time t, the number of partitions may depend on (n_t, n, T). Let $J_t^{d_z}$ be the number of partitions for time t and by assumption we have that, uniformly in t, $J_t \asymp J$ for some sequence $J = J_{n,T} \to \infty$ as $n, T \to \infty$. Throughout the Appendix, for simplicity of notation, we will suppress any dependence and just refer to J_t and J.

If we write

$$\mu_t^0(z) = B_t(z)' \gamma_t^0, \qquad B_t(z) = \left(\widehat{\mathbb{I}}_1(z), \ldots, \widehat{\mathbb{I}}_{J_t^{d_z}}(z)\right)',$$

where

$$\widehat{\mathbb{I}}_{jt}(z) = \widehat{\mathbb{I}}_{j_1t,1}(z_1)\,\widehat{\mathbb{I}}_{j_2t,2}(z_2)\cdots\widehat{\mathbb{I}}_{j_dt,d}(z_d),$$

and

$$\begin{aligned}
\left\{\widehat{\mathbb{I}}_{j_\ell t,\ell}(z_\ell) = 1\right\} \quad &\Longleftrightarrow \quad \left\{\hat{b}_{(j_\ell-1)t,\ell} \le z_\ell < \hat{b}_{j_\ell t,\ell}\right\}, \qquad 1 \le j_\ell < J,\\
\left\{\widehat{\mathbb{I}}_{j_\ell t,\ell}(z_\ell) = 1\right\} \quad &\Longleftrightarrow \quad \left\{\hat{b}_{(j_\ell-1)t,\ell} \le z_\ell \le \hat{b}_{j_\ell t,\ell}\right\} \qquad j_\ell = J,
\end{aligned}$$

where $\hat{b}_{j_\ell,\ell} = \hat{F}_{t,\ell}^{-1}(j_\ell/J)$ and $\hat{F}_{t,\ell}^{-1}(\cdot)$ is the empirical quantile function of $z_{it,\ell}$ for the cross-section at time t. Our estimator of $\mu(\cdot)$ at time t is

$$\hat{\mu}_t(z) = B_t(z)'\hat{\gamma}_t, \qquad \hat{\gamma}_t = \left(B_t' M_{X_t} B_t\right)^{-1} B_t' M_{X_t} R_t,$$

where $B_t = (B_t(z_{1t}), \ldots, B_t(z_{n_t t}))'$ is $n_t \times J_t^{d_z}$, $M_{X_t} = I_{n_t} - X_t(X_t'X_t)^{-1}X_t'$, and X_t is the $n_t \times d_x$ matrix of the stacked x_{it}s.

Furthermore our estimator of $\mu(\cdot)$ based on the full sample is

$$\hat{\mu}(z) = \frac{1}{T}\sum\nolimits_{t=1}^{T} \hat{\mu}_t(z).$$

Corresponding to $\hat{\mu}_t(z)$, our estimator of β_t at time t is,

$$\hat{\beta}_t = \left(X_t' M_{B_t} X_t\right)^{-1} X_t' M_{B_t} R_t,$$

where $M_{B_t} = I_{n_t} - B_t(B_t'B_t)^{-1}B_t'$.

It will also be useful to introduce some additional definitions. First, let $\mathbb{I}_{jt}(z)$ be defined similar as above so that

$$\mathbb{I}_{jt}(z) = \mathbb{I}_{j_1t,1}(z_1)\,\mathbb{I}_{j_2t,2}(z_2)\cdots\mathbb{I}_{j_dt,d}(z_d),$$

and

$$\begin{aligned}
\{\mathbb{I}_{j_\ell t,\ell}(z_\ell) = 1\} \quad &\Longleftrightarrow \quad \left\{b_{(j_\ell-1)t,\ell} \le z_\ell < b_{j_\ell t,\ell}\right\}, \qquad 1 \le j_\ell < J,\\
\{\mathbb{I}_{j_\ell t,\ell}(z_\ell) = 1\} \quad &\Longleftrightarrow \quad \left\{b_{(j_\ell-1)t,\ell} \le z_\ell \le b_{j_\ell t,\ell}\right\} \qquad j_\ell = J,
\end{aligned}$$

and $b_{j_\ell,\ell} = F_{t,\ell}^{-1}(j_\ell/J)$ and $F_{t,\ell}^{-1}(\cdot)$ is the quantile function for $z_{it,\ell}$ for the cross-section at time t. Then, recall that,

$$q_{jt} = \mathbb{E}\left(\mathbb{I}_{jt}(z_{it}) \middle| \mathcal{F}_t\right).$$

The sample analog is

$$\tilde{q}_{jt} = \frac{1}{n_t} \sum\nolimits_{i=1}^{n_t} \mathbb{I}_{jt} \left(z_{it} \right).$$

Further define

$$\hat{q}_{jt} = \frac{1}{n_t} \sum\nolimits_{i=1}^{n_t} \widehat{\mathbb{I}}_{jt} \left(z_{it} \right).$$

It will also be useful to define 1_{jt} as

$$\mathbf{1}_{jt} = \mathbf{1}_{q,jt} \mathbf{1}_{\beta,t} = \mathbf{1} \left\{ \hat{q}_{jt} \geq q_{jt}/\, 2 \right\} \times \mathbf{1} \left\{ \lambda_{\min} \left(\hat{\Omega}_{\mathrm{uu,t}} \right) \geq C_{\mathrm{uu}}/\, 2 \right\},$$

where C_{uu} is the lower bound, uniformly in t, introduced in Assumption 1(2).

Finally define

$$\begin{aligned} \mathrm{V}(z) &= T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z)\, \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it})\, \sigma_{it}^2, \\ \tilde{\mathrm{V}}(z) &= T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z)\, \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it})\, \varepsilon_{it}^2 \end{aligned}$$

along with

$$\begin{aligned} \hat{\mathrm{V}}_{\mathrm{FM}}(z) &= T^{-2} \sum\nolimits_{t=1}^{T} \left(\hat{\mu}_t(z) - \hat{\mu}(z) \right)^2 \\ \hat{\mathrm{V}}_{\mathrm{PI}}(z) &= T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z)\, \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it})\, \hat{\varepsilon}_{it}^2. \end{aligned}$$

B.4 Lemmas

Our first lemma is a generalization of Cattaneo and Farrell (2013, Lemma A.2) to allow for random partitions.

Lemma 1. *Let Assumptions 1-3 hold. Then, (i) there exists a γ_{jt}^0 such that*

$$\max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t^{d_z}} \sup_z \left| \widehat{\mathbb{I}}_{jt}(z)\, \mu(z) - \widehat{\mathbb{I}}_{jt}(z)\, \gamma_{jt}^0 \right| = O_p\left(J^{-1} \right),$$

and

$$\mathbb{E}\left[\max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t^{d_z}} \sup_z \left| \widehat{\mathbb{I}}_{jt}(z)\, \mu(z) - \widehat{\mathbb{I}}_{jt}(z)\, \gamma_{jt}^0 \right|^2 \right] = O\left(J^{-2} \right).$$

(ii) If we define $h_{t,\ell}(z) = h_{t,\ell}(z, \mathcal{F}_t) = E\left[x_{it,\ell} | \mathcal{F}_t, z_{it} = z \right]$ where $x_{it,\ell}$ is the ℓth element of x_{it}, there exists a $\pi_{jt,\ell}^0$ such that

$$\max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t^{d_z}} \sup_z \left| \widehat{\mathbb{I}}_{jt}(z)\, h_{t,\ell}(z) - \widehat{\mathbb{I}}_{jt}(z)\, \pi_{jt,\ell}^0 \right| = O_p\left(J^{-1} \right),$$

and

$$\mathbb{E}\left[\max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t^{d_z}} \sup_z \left| \widehat{\mathbb{I}}_{jt}(z)\, h_{t,\ell}(z) - \widehat{\mathbb{I}}_{jt}(z)\, \pi_{jt,\ell}^0 \right|^2 \right] = O\left(J^{-2} \right).$$

Stack $\gamma_t^0 = \left(\gamma_1^0, \ldots, \gamma_{J_t^{d_z}}^0\right)'$ and

$$\Pi_t^0 = \begin{bmatrix} \pi_{1t,1}^0 & \cdots & \pi_{1t,d_x}^0 \\ \vdots & \ddots & \vdots \\ \pi_{J_t^{d_z}t,1}^0 & \cdots & \pi_{J_t^{d_z}t,d_x}^0 \end{bmatrix}$$

We also stack the $h_{t,\ell}(\cdot)$'s as $h_t(\cdot) = (h_{t,1}(\cdot), \ldots, h_{t,d_x}(\cdot))'$ and then stack again as the $n_t \times d_x$ matrix $H_t = (h_t(z_{1t}), \ldots, h_t(z_{n_t t}))'$. Finally, define $U_t = X_t - H_t$. Recall from above that $\Omega_{\mathrm{uu},t} = \mathrm{plim}_{n\to\infty} U_t U_t'/n_t = \mathbb{E}\left[(x_{it} - \mathbb{E}[x_{it}|\, z_{it}, \mathcal{F}_t])(x_{it} - \mathbb{E}[x_{it}|\, z_{it}, \mathcal{F}_t])' \middle|\, \mathcal{F}_t\right] = \mathbb{E}[\mathbb{V}(x_{it}|\, z_{it}, \mathcal{F}_t)|\, \mathcal{F}_t]$.

Lemma 2. *Let Assumptions 1-3 hold. Then,*

$$\max_{1\le t\le T} \max_{1\le j\le J_t^{d_z}} |\hat{q}_{jt} - q_{jt}|^2 = O_p\left(\frac{\log\left(J^{d_z} \vee T\right)}{J^{d_z} n}\right).$$

Lemma 3. *Let Assumptions 1-3 hold. Define,*

$$\hat{\Omega}_{\mathrm{uu},t} = X_t' M_{B_t} X_t /\, n_t.$$

Then,

$$\frac{1}{T}\sum\nolimits_{t=1}^{T} \left\|\hat{\Omega}_{\mathrm{uu},t} - \Omega_{\mathrm{uu},t}\right\|^2 = O_p\left(n^{-1}\right) + O_p\left(J^{-4}\right) + O_p\left(n^{-2} J^{2d_z}\right),$$

and

$$\max_{1\le t\le T} \left\|\hat{\Omega}_{\mathrm{uu},t} - \Omega_{\mathrm{uu},t}\right\| = O_p\left(\log(T)\, n^{-1/2}\right) + O_p\left(J^{-2}\right) + O_p\left(n^{-1} J^{d_z}\right).$$

Lemma 4. *Let Assumptions 1-3 hold. Then,*

$$\left|T^{-1}\sum\nolimits_{t=1}^{T} \mathbf{1}_{\beta,t} s_t' \left(\hat{\beta}_t - \beta_t\right)\right|^2 = O_p\left(n^{-1}T^{-1}\right) + O_p\left(J^{-4}\right) + O_p\left(J^{2d_z} n^{-3}\right) + O_p\left(J^{d_z - 4} n^{-2}\right)$$

and

$$T^{-1}\sum\nolimits_{t=1}^{T} \mathbf{1}_{\beta,t}\left(s_t'\left(\hat{\beta}_t - \beta_t\right)\right)^2 = O_p\left(n^{-1}\right) + O_p\left(J^{-4}\right),$$

where $\|s_t\| \le C$ almost surely and s_t is nonrandom conditional on z_t and $\mathcal{F}_t$.

Lemma 5. *Let Assumptions 1-3 hold. Then, $C_1 n^{-1} T^{-1} J^{d_z}[1 + o_p(1)] \le \mathrm{V}(z) \le C_2 n^{-1} T^{-1} J^{d_z}[1 + o_p(1)]$ for constants C_1 and C_2 bounded and bounded away from zero.*

Lemma 6. *Under Assumptions 1-3 and if $J^d \log(T)^2 \log\left(T \wedge J^d\right)^{-1} = O(n)$ then*

$$\mathrm{V}(z)^{-1} T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z)\, \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it}) \left(\varepsilon_{it}^2 - \sigma_{it}^2\right) = o_p(1).$$

Before proceeding note that by Lemma 2 and 3 we have that $\mathbf{1}_{q,jt} \to 1$ and $\mathbf{1}_{\beta,t} \to 1$ with probability approaching one.

B.5 Proof of Theorem 1

Recall that our estimator is

$$\hat{\mu}(z) = T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left(R_{it} - x_{it}'\hat{\beta}_t\right),$$

which can be decomposed as

$$\hat{\mu}(z) - \mu(z) = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4,$$

where

$$\begin{aligned}
\mathcal{L}_1 &= T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left(\mu(z_{it}) - \mu(z)\right),\\
\mathcal{L}_2 &= T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\,\varepsilon_{it},\\
\mathcal{L}_3 &= -T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\,x_{it}'\left(\hat{\beta}_t - \beta_t\right),\\
\mathcal{L}_4 &= T^{-1}\sum_{t=1}^{T}\sum_{j=1}^{J_t^{d_z}} \left(\mathbf{1}_{jt} - 1\right)\widehat{\mathbb{I}}_{jt}(z)\,\mu(z).
\end{aligned}$$

We will work with the re-scaled estimator:

$$\mathrm{V}(z)^{-1/2}\left(\hat{\mu}(z) - \mu(z)\right) = \mathrm{V}(z)^{-1/2}\mathcal{L}_1 + \mathrm{V}(z)^{-1/2}\mathcal{L}_2 + \mathrm{V}(z)^{-1/2}\mathcal{L}_3 + \mathrm{V}(z)^{-1/2}\mathcal{L}_4,$$

and show that

$$\mathrm{V}(z)^{-1/2}\left(\hat{\mu}(z) - \mu(z)\right) = \mathrm{V}(z)^{-1/2}\mathcal{L}_2 + o_p(1), \qquad \mathrm{V}(z)^{-1/2}\mathcal{L}_2 \longrightarrow_d \mathcal{N}(0,1).$$

B.5.1 Term: $\mathcal{L}_1$

By Lemma 5 we need only show that

$$\left|J^{-d/2}T^{-1/2}\sum_{t=1}^{T} n^{1/2}n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left(\mu(z_{it}) - \mu(z)\right)\right| = o_p(1).$$

We have,

$$\begin{aligned}
&\left|T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left(\mu(z_{it}) - \mu(z)\right)\right|\\
\leq\ & T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left|\mu(z_{it}) - \gamma_{jt}^{0}\right|\\
&+T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left|\mu(z) - \gamma_{jt}^{0}\right|.
\end{aligned}$$

The first term is

$$\begin{aligned}
&T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left|\mu(z_{it}) - \gamma_{jt}^{0}\right|\\
\leq\ & \max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\sup_{z}\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z) - \widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^{0}\right| \times T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\\
\leq\ & \max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\sup_{z}\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z) - \widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^{0}\right|,
\end{aligned}$$

which is $O_p\left(J^{-1}\right)$ by Lemma 1. The second term follows by the same steps so that

$$\left|J^{-d/2}T^{-1/2}\sum\nolimits_{t=1}^{T}n^{1/2}n_t^{-1}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}\left(z\right)\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}\left(z_{it}\right)\left(\mu\left(z_{it}\right)-\mu\left(z\right)\right)\right|=O_p\left(J^{-(d/2+1)}T^{1/2}n^{1/2}\right),$$

which is $o_p(1)$ under our rate assumptions.

B.5.2 Term: $\mathcal{L}_2$

For $\mathcal{L}_2$ define the sigma field, $\mathcal{G}_s=\sigma\left(z_1,\ldots,z_T,x_1,\ldots,x_T,\mathcal{F}_1,\ldots,\mathcal{F}_T,\varepsilon_1,\ldots\varepsilon_s\right)$ and the variable

$$\xi_s=\mathrm{V}\left(z\right)^{-1/2}T^{-1}n_s^{-1}\sum\nolimits_{i=1}^{n_s}\sum\nolimits_{j=1}^{J_s^{dz}}\mathbf{1}_{js}\widehat{\mathbb{I}}_{js}\left(z\right)\hat{q}_{js}^{-1}\widehat{\mathbb{I}}_{js}\left(z_{is}\right)\varepsilon_{is}.$$

Note first that

$$\begin{aligned}
&\sum\nolimits_{t=1}^{T}\mathbb{E}\left[\left.\left|\xi_t^2\right|\mathcal{G}_{t-1}\right]\\
=\;&\mathrm{V}\left(z\right)^{-1}T^{-2}\sum\nolimits_{t=1}^{T}n_t^{-2}\sum\nolimits_{i_1=1}^{n_t}\sum\nolimits_{i_2=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}\left(z\right)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}\left(z_{i_1t}\right)\widehat{\mathbb{I}}_{jt}\left(z_{i_2t}\right)\times\\
&\qquad\mathbb{E}\left[\left.\mathbb{E}\left[\left.\varepsilon_{i_1t}\varepsilon_{i_2t}\right|\mathcal{F}_t,z_t,x_t,\mathcal{G}_{t-1}\right]\right|\mathcal{G}_{t-1}\right]\\
=\;&\mathrm{V}\left(z\right)^{-1}T^{-2}\sum\nolimits_{t=1}^{T}n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}\left(z\right)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}\left(z_{it}\right)\sigma_{it}^2\\
=\;&1.
\end{aligned}$$

Clearly $\xi_s\in\mathcal{G}_s$ and

$$\mathbb{E}\left[\left.\xi_s\right|\mathcal{G}_{s-1}\right]=\mathrm{V}\left(z\right)^{-1/2}T^{-1}n_s^{-1}\sum\nolimits_{i=1}^{n_s}\sum\nolimits_{j=1}^{J_s^{dz}}\mathbf{1}_{js}\widehat{\mathbb{I}}_{js}\left(z\right)\hat{q}_{js}^{-1}\widehat{\mathbb{I}}_{js}\left(z_{is}\right)\mathbb{E}\left[\left.\varepsilon_{is}\right|\mathcal{G}_{s-1}\right],$$

with $\mathbb{E}\left[\left.\varepsilon_{is}\right|\mathcal{G}_{s-1}\right]=0$. Thus, $(\xi_s,\mathcal{G}_s)$ is a martingale difference sequence with $\sum_{t=1}^{T}\mathbb{E}\left[\left.\left|\xi_t^2\right|\mathcal{G}_{t-1}\right]=1$. By Hall and Heyde (1980, Corollary 3.1) we need only show that

$$\sum\nolimits_t\mathbb{E}\left[\left.\xi_t^2\mathbf{1}\left\{\left|\xi_t\right|>\epsilon\right\}\right|\mathcal{G}_{t-1}\right]=o_p(1)\qquad\text{for all }\epsilon>0.$$

This is implied by showing that

$$\sum\nolimits_t\mathbb{E}\left[\left.\left|\xi_t\right|^{2+\delta}\right|\mathcal{G}_{t-1}\right]=o_p(1),$$

for some $\delta>0$. To show this note that,

$$\begin{aligned}
&\sum\nolimits_t\mathbb{E}\left[\left.\left|\xi_t\right|^{2+\delta}\right|\mathcal{G}_{t-1}\right]\\
=\;&\mathrm{V}\left(z\right)^{-(1+\delta/2)}T^{-(2+\delta)}\sum\nolimits_t\mathbb{E}\left[\left.\mathbb{E}\left[\left.\left|n_t^{-1}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}\left(z\right)\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}\left(z_{it}\right)\varepsilon_{it}\right|^{2+\delta}\right|\mathcal{F}_t,x_t,z_t,\mathcal{G}_{t-1}\right]\right|\mathcal{G}_{t-1}\right].
\end{aligned}$$

Then note that

$$\begin{aligned}
&\mathbb{E}\left[\left.\left|\sum\nolimits_{i=1}^{n_t}n_t^{-1}\sum\nolimits_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}\left(z\right)\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}\left(z_{it}\right)\varepsilon_{it}\right|^{2+\delta}\right|\mathcal{F}_t,x_t,z_t,\mathcal{G}_{t-1}\right]\\
\leq\;&C\sum\nolimits_{i=1}^{n_t}\mathbb{E}\left[\left.\left|n_t^{-1}\sum\nolimits_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}\left(z\right)\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}\left(z_{it}\right)\varepsilon_{it}\right|^{2+\delta}\right|\mathcal{F}_t,x_t,z_t,\mathcal{G}_{t-1}\right]\vee
\end{aligned}$$

$$C\left(\sum_{i=1}^{n_t}\mathbb{E}\left[\left|n_t^{-1}\sum_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\,\varepsilon_{it}\right|^2\middle|\mathcal{F}_t,x_t,z_t,\mathcal{G}_{t-1}\right]\right)^{1+\delta/2}.$$

The first term is

$$\begin{aligned}
&\sum_{i=1}^{n_t}\mathbb{E}\left[\left|n_t^{-1}\sum_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\,\varepsilon_{it}\right|^{2+\delta}\middle|\mathcal{F}_t,x_t,z_t,\mathcal{G}_{t-1}\right]\\
\leq\ & C\sum_{i=1}^{n_t}\left|n_t^{-1}\sum_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\right|^{2+\delta}\\
\leq\ & Cn^{-(2+\delta)}J^{d(2+\delta)}\sum_{i=1}^{n_t}\left|\sum_{j=1}^{J_t^{dz}}\widehat{\mathbb{I}}_{jt}(z)\,\widehat{\mathbb{I}}_{jt}(z_{it})\right|^{2+\delta}\\
=\ & Cn^{-(2+\delta)}J^{d(2+\delta)}\sum_{j=1}^{J_t^{dz}}\widehat{\mathbb{I}}_{jt}(z)\sum_{i=1}^{n_t}\widehat{\mathbb{I}}_{jt}(z_{it})\\
=\ & C\left(n^{-1}J^d\right)^{1+\delta}
\end{aligned}$$

The second term is

$$\begin{aligned}
&\left(\sum_{i=1}^{n_t}\mathbb{E}\left[\left|n_t^{-1}\sum_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\,\varepsilon_{it}\right|^2\middle|\mathcal{F}_t,x_t,z_t,\mathcal{G}_{t-1}\right]\right)^{1+\delta/2}\\
\leq\ & C\left(J^{2d}n_t^{-2}\sum_{i=1}^{n_t}\left|\sum_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\widehat{\mathbb{I}}_{jt}(z_{it})\right|^2\right)^{1+\delta/2}\\
=\ & C\left(J^dn^{-1}\right)^{1+\delta/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_t\mathbb{E}\left[|\xi_t|^{2+\delta}\middle|\mathcal{G}_{t-1}\right]\\
\leq\ & C\left(J^dn^{-1}T^{-1}\right)^{-(1+\delta/2)}T^{-(1+\delta)}\left(n^{-(1+\delta)}J^{d(1+\delta)}\vee\left(J^dn^{-1}\right)^{1+\delta/2}\right)\\
\leq\ & CT^{-\delta/2},
\end{aligned}$$

and the result follows.

B.5.3 Term: $\mathcal{L}_3$

We have

$$\begin{aligned}
-\mathrm{V}(z)^{-1/2}\mathcal{L}_3 &= \mathrm{V}(z)^{-1/2}T^{-1}\sum_{t=1}^{T}n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{dz}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\,x_{it}'\left(\hat{\beta}_t-\beta_t\right)\\
&= \mathrm{V}(z)^{-1/2}T^{-1}\sum_{t=1}^{T}\mathbf{1}_{\beta,t}\hat{h}_t(z)'\left(\hat{\beta}_t-\beta_t\right),
\end{aligned}$$

where

$$\hat{h}_t(z)=\sum_{j=1}^{J_t^{dz}}\widehat{\mathbb{I}}_{jt}(z)\,\hat{\pi}_{jt},\qquad \hat{\pi}_{jt}=\mathbf{1}_{q,jt}\hat{q}_{jt}^{-1}n_t^{-1}\sum_{i=1}^{n_t}\widehat{\mathbb{I}}_{jt}(z_{it})\,x_{it}.$$

Thus,

$$-\mathrm{V}(z)^{-1/2}\mathcal{L}_3 = \mathrm{V}(z)^{-1/2}T^{-1}\sum_{t=1}^{T}\mathbf{1}_{\beta,t}h_t(z)'\left(\hat{\beta}_t-\beta_t\right)$$

$$
\begin{aligned}
&+\mathrm{V}(z)^{-1/2} T^{-1} \sum_{t=1}^{T} \mathbf{1}_{\beta,t} \left(\hat{h}_t(z) - h_t(z)\right)' \left(\hat{\beta}_t - \beta_t\right) \\
=\ & \mathrm{V}(z)^{-1/2} \mathcal{L}_{31} + \mathrm{V}(z)^{-1/2} \mathcal{L}_{32}.
\end{aligned}
$$

First we have,

$$
\begin{aligned}
|\mathcal{L}_{31}|^2 &= \left| T^{-1} \sum_{t=1}^{T} \mathbf{1}_{\beta,t} h_t(z)' \left(\hat{\beta}_t - \beta_t\right) \right|^2 \\
&= O_p\left(n^{-1}T^{-1}\right) + O_p\left(J^{-4}\right) + O_p\left(J^{2d_z} n^{-3}\right) + O_p\left(J^{d_z-4} n^{-2}\right),
\end{aligned}
$$

by Lemma 4. Thus,

$$
\frac{nT}{J^d} |\mathcal{L}_{31}|^2 = O_p\left(J^{-d_z}\right) + O_p\left(\frac{nT}{J^{d_z+2}} J^{-2}\right) + O_p\left(\frac{J^{d_z}}{n} \frac{T}{n}\right) + O_p\left(\frac{T}{nJ^4}\right),
$$

which is $o_p(1)$ under Assumption 3. Next, by the Cauchy-Schwartz inequality

$$
|\mathcal{L}_{32}|^2 \leq \frac{1}{T} \sum_{t=1}^{T} \left\| \hat{h}_t(z) - h_t(z) \right\|^2 \times \frac{1}{T} \sum_{t=1}^{T} \mathbf{1}_{\beta,t} \left\| \hat{\beta}_t - \beta_t \right\|^2.
$$

The order of the first factor follows by exactly the same steps as in the proof of Theorem 2 for the consistency of the Fama-MacBeth style variance estimator (ignoring the $\mathcal{S}_{12}^{\mathrm{FM}}$ term). That is, we show below that

$$
\frac{1}{T^2} \sum_{t=1}^{T} \left(\hat{\mu}(z) - \mu(z)\right)^2 = O_p\left(\frac{J^d}{nT}\right).
$$

Thus, the first factor is $O_p\left(n^{-1}J^d\right)$. By Lemma 4, the second factor is $O_p\left(n^{-1}\right) + O_p\left(J^{-4}\right)$. Thus,

$$
\frac{nT}{J^d} |\mathcal{L}_{32}|^2 = O_p\left(n^{-1}T\right) + O_p\left(TJ^{-4}\right),
$$

which is $o_p(1)$ under our assumptions.

B.5.4 Term: $\mathcal{L}_4$

Finally consider $\mathcal{L}_4$:

$$
\begin{aligned}
\left| \mathrm{V}(z)^{-1/2} \mathcal{L}_4 \right| &= \left| \mathrm{V}(z)^{-1/2} T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{J_t^{d_z}} \left(\mathbf{1}_{jt} - 1\right) \widehat{\mathbb{I}}_{jt}(z) \mu(z) \right| \\
&\leq C\mathrm{V}(z)^{-1/2} T^{-1} \sum_{t=1}^{T} \sum_{j=1}^{J_t^{d_z}} \left(|\mathbf{1}_{jt} - 1|\right) \widehat{\mathbb{I}}_{jt}(z) \\
&\leq C n^{1/2} T^{1/2} J^{-d/2} \times \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t^{d_z}} |\mathbf{1}_{jt} - 1|,
\end{aligned}
$$

Thus, $\left| \mathrm{V}(z)^{-1/2} \mathcal{L}_4 \right| = o_p(1)$ by Lemmas 2 and 3.

B.6 Proof of Theorem 2

B.6.1 Proof of Consistency of Variance Estimators (FM-style Variance Estimator)

We need to show that

$$\frac{nT}{J^{d_z}}\left(\hat{\mathrm{V}}_{\mathrm{FM}}\left(z\right)-\mathrm{V}\left(z\right)\right)=o_p\left(1\right),\qquad \hat{\mathrm{V}}_{\mathrm{FM}}\left(z\right)=T^{-2}\sum\nolimits_{t=1}^{T}\left(\hat{\mu}_t\left(z\right)-\hat{\mu}\left(z\right)\right)^2.$$

First note that

$$\hat{\mathrm{V}}_{\mathrm{FM}}\left(z\right)=T^{-2}\sum\nolimits_{t=1}^{T}\left(\hat{\mu}_t\left(z\right)-\hat{\mu}\left(z\right)\right)^2=T^{-2}\sum\nolimits_{t=1}^{T}\left(\hat{\mu}_t\left(z\right)-\mu\left(z\right)\right)^2-T^{-1}\left(\hat{\mu}\left(z\right)-\mu\left(z\right)\right)^2.$$

Recall that,

$$\begin{aligned}
\hat{\mu}_{t_1}\left(z\right)-\mu\left(z\right) &= n_{t_1}^{-1}\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\sum\nolimits_{i_1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z_{i_1t_1}\right)\left(R_{i_1t_1}-x_{i_1t_1}'\hat{\beta}_{t_1}\right)-\mu\left(z\right)\\
&= n_{t_1}^{-1}\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\sum\nolimits_{i_1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z_{i_1t_1}\right)\varepsilon_{i_1t_1}\\
&\quad +n_{t_1}^{-1}\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\sum\nolimits_{i_1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z_{i_1t_1}\right)\left(\mu\left(z_{i_1t_1}\right)-\mu\left(z\right)\right)\\
&\quad -n_{t_1}^{-1}\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\sum\nolimits_{i_1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z_{i_1t_1}\right)x_{i_1t_1}'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\\
&\quad +\sum\nolimits_{j_1}\left(\mathbf{1}_{j_1t_1}-1\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\mu\left(z\right).
\end{aligned}$$

Thus, since we have already shown that $\frac{nT}{J^{d_z}}\left(\tilde{\mathrm{V}}\left(z\right)-\mathrm{V}\left(z\right)\right)=o_p\left(1\right)$ by Lemma 6 then by the CS inequality it is sufficient to show that $\left|\mathcal{S}_{11}^{\mathrm{FM}}\right|=o_p\left(1\right)$, $\left|\mathcal{S}_{12}^{\mathrm{FM}}\right|=o_p\left(1\right)$, $\left|\mathcal{S}_{13}^{\mathrm{FM}}\right|=o_p\left(1\right)$, and $\left|\mathcal{S}_{2}^{\mathrm{FM}}\right|=o_p\left(1\right)$ where

$$\begin{aligned}
\mathcal{S}_{11}^{\mathrm{FM}} &= \frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\left[n_{t_1}^{-1}\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\sum\nolimits_{i_1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z_{i_1t_1}\right)\left(\mu\left(z_{i_1t_1}\right)-\mu\left(z\right)\right)\right]^2\\
\mathcal{S}_{12}^{\mathrm{FM}} &= \frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\left[n_{t_1}^{-1}\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\sum\nolimits_{i_1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z_{i_1t_1}\right)x_{i_1t_1}'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\right]^2\\
\mathcal{S}_{13}^{\mathrm{FM}} &= \frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\left[\sum\nolimits_{j_1}\left(\mathbf{1}_{j_1t_1}-1\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\mu\left(z\right)\right]^2\\
\mathcal{S}_{2}^{\mathrm{FM}} &= \frac{n}{J^{d_z}}\left(\hat{\mu}\left(z\right)-\mu\left(z\right)\right)^2.
\end{aligned}$$

First consider, $\mathcal{S}_2^{\mathrm{FM}}$. We have already shown that $\hat{\mu}\left(z\right)-\mu\left(z\right)=O_p\left(\sqrt{J^{d_z}n^{-1}T^{-1}}\right)$, so that $\mathcal{S}_2$ satisfies

$$\mathcal{S}_2^{\mathrm{FM}}=\frac{n}{J^{d_z}}\left(\hat{\mu}\left(z\right)-\mu\left(z\right)\right)^2=O_p\left(T^{-1}\right)=o_p\left(1\right).$$

Next, consider $\mathcal{S}_{11}^{\mathrm{FM}}$,

$$\begin{aligned}
&\mathcal{S}_{11}^{\mathrm{FM}}\\
=\;& \frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\left[n_{t_1}^{-1}\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\sum\nolimits_{i_1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z_{i_1t_1}\right)\left(\mu\left(z_{i_1t_1}\right)-\mu\left(z\right)\right)\right]^2\\
\leq\;& \frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\left[n_{t_1}^{-1}\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\sum\nolimits_{i_1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)\widehat{\mathbb{I}}_{j_1t_1}\left(z_{i_1t_1}\right)\left(\left|\mu\left(z_{i_1t_1}\right)-\gamma_{j_1t_1}^0\right|+\left|\mu\left(z\right)-\gamma_{j_1t_1}^0\right|\right)\right]^2\\
\leq\;& C\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}\left(z\right)\mu\left(z\right)-\widehat{\mathbb{I}}_{jt}\left(z\right)\gamma_{j_1t_1}^0\right|^2\times\frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\left[\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z\right)n_{t_1}^{-1}\sum\nolimits_{i_1}\widehat{\mathbb{I}}_{j_1t_1}\left(z_{i_1t_1}\right)\right]^2\\
\leq\;& C\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}\left(z\right)\mu\left(z\right)-\widehat{\mathbb{I}}_{jt}\left(z\right)\gamma_{j_1t_1}^0\right|^2\times\frac{n}{J^{d_z}},
\end{aligned}$$

and so $\mathcal{S}_{11}^{\text{FM}} = O_p\left(nJ^{-(d_z+2)}\right)$ which is $o(1)$ under our rate assumptions.

Next consider $\mathcal{S}_{12}^{\text{FM}}$,

$$\begin{aligned}
\mathcal{S}_{12}^{\text{FM}} &= \frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\left[n_{t_1}^{-1}\sum\nolimits_{j_1}\hat{q}_{j_1t_1}^{-1}\sum\nolimits_{i_1}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}(z)\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})x_{i_1t_1}'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\right]^2 \\
&= \frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\left[\mathbf{1}_{\beta,t_1}\left(\hat{h}_{t_1}(z)-h_{t_1}(z)\right)'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)+\mathbf{1}_{\beta,t_1}h_{t_1}(z)'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\right]^2 \\
&\leq C\frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\mathbf{1}_{\beta,t_1}\left[h_{t_1}(z)'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\right]^2+C\frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\mathbf{1}_{\beta,t_1}\left[\left(\hat{h}_{t_1}(z)-h_{t_1}(z)\right)'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\right]^2 \\
&= \mathcal{S}_{121}^{\text{FM}}+\mathcal{S}_{122}^{\text{FM}}.
\end{aligned}$$

$\mathcal{S}_{121}^{\text{FM}}$ follows by exactly the same steps as in the proof for $\mathcal{L}_3$ so that we have

$$\begin{aligned}
\frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\mathbf{1}_{\beta,t_1}\left[h_t(z)'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\right]^2 &= \frac{n}{J^{d_z}}\cdot T\cdot\left(O_p\left(n^{-1}T^{-1}\right)+O_p\left(T^{-1}J^{-4}\right)\right) \\
&= O_p\left(J^{-d_z}\right)+O_p\left(\frac{nT}{J^{d_z+2}}\times\frac{1}{TJ^2}\right),
\end{aligned}$$

which is $o_p(1)$ under our Assumptions 3. Next consider $\mathcal{S}_{122}^{\text{FM}}$,

$$\begin{aligned}
&\frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\mathbf{1}_{\beta,t_1}\left|\left(\hat{h}_t(z)-h_t(z)\right)'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\right|^2 \\
\leq\ &\left(\left(\frac{n}{J^{d_z}}\right)T^{-1}\sum\nolimits_{t_1}\left\|\hat{h}_t(z)-h_t(z)\right\|^2\right)\left(\sum\nolimits_{t_1}\mathbf{1}_{\beta,t_1}\left\|\hat{\beta}_{t_1}-\beta_{t_1}\right\|^2\right).
\end{aligned}$$

The first factor is $O_p(1)$. To see this note that we can show that $\left(\frac{n}{J^{d_z}}\right)T^{-1}\sum_{t_1}\left(\hat{\mu}_t(z)-\mu(z)\right)^2 = O_p(1)$ by showing $\left|\mathcal{S}_{11}^{\text{FM}}\right| = o_p(1)$, $\left|\mathcal{S}_{12}^{\text{FM}}\right| = o_p(1)$, and $\left|\mathcal{S}_{2}^{\text{FM}}\right| = o_p(1)$. We can then follow the same steps to show that $\left(\frac{n}{J^{d_z}}\right)T^{-1}\sum_{t_1}\left\|\hat{h}_t(z)-h_t(z)\right\|^2 = O_p(1)$. The second factor is $O_p\left(Tn^{-1}\right)+O_p\left(TJ^{-4}\right)$ which is $o_p(1)$ by our rate assumptions and since $TJ^{-4} = TJ^{-2}\cdot J^{-2}$ which is $o(1)$ under Assumption 3.

Finally consider $\mathcal{S}_{13}^{\text{FM}}$,

$$\begin{aligned}
\mathcal{S}_{13}^{\text{FM}} &= \frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\left[\sum\nolimits_{j_1}\left(\mathbf{1}_{j_1t_1}-1\right)\widehat{\mathbb{I}}_{j_1t_1}(z)\mu(z)\right]^2 \\
&= \frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\sum\nolimits_{j_1}\left|\mathbf{1}_{j_1t_1}-1\right|\widehat{\mathbb{I}}_{j_1t_1}(z)\mu(z)^2 \\
&\leq \sup_z\mu(z)^2\times\frac{n}{TJ^{d_z}}\sum\nolimits_{t_1}\sum\nolimits_{j_1}\left|\mathbf{1}_{j_1t_1}-1\right|\widehat{\mathbb{I}}_{j_1t_1}(z) \\
&\leq C\left(\frac{n}{J^{d_z}}\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\left|\mathbf{1}_{jt}-1\right|\right),
\end{aligned}$$

so that $\mathcal{S}_{13}^{\text{FM}} = o_p(1)$.

B.6.2 Proof of Consistency of Variance Estimators (Plug-in Variance Estimator)

We need to show that

$$\frac{nT}{J^{d_z}}\left(\hat{\text{V}}_{\text{PI}}(z)-\text{V}(z)\right) = o_p(1), \qquad \hat{\text{V}}_{\text{PI}}(z) = T^{-2}\sum\nolimits_{t=1}^{T}n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\hat{\varepsilon}_{it}^2.$$

First note that

$$\frac{nT}{J^{d_z}}\left(\hat{\mathrm{V}}_{\mathrm{PI}}(z)-\mathrm{V}(z)\right)=\frac{nT}{J^{d_z}}\left(\tilde{\mathrm{V}}(z)-\mathrm{V}(z)\right)+\mathcal{S}_1^{\mathrm{PI}}+\mathcal{S}_2^{\mathrm{PI}}+\mathcal{S}_3^{\mathrm{PI}},$$

where

$$\begin{aligned}
\mathcal{S}_1^{\mathrm{PI}} &= -\frac{n}{J^{d_z}T}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i_1\neq i_2}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{i_1t})\,\widehat{\mathbb{I}}_{jt}(z_{i_2t})\,\varepsilon_{i_1t}\varepsilon_{i_2t}\\
\mathcal{S}_2^{\mathrm{PI}} &= \frac{2n}{J^{d_z}T}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\,\varepsilon_{it}\left(\hat{\varepsilon}_{it}-\varepsilon_{it}\right)\\
\mathcal{S}_3^{\mathrm{PI}} &= \frac{n}{J^{d_z}T}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\left(\hat{\varepsilon}_{it}-\varepsilon_{it}\right)^2.
\end{aligned}$$

Again, we have already shown that $\frac{nT}{J^{d_z}}\left(\tilde{\mathrm{V}}(z)-\mathrm{V}(z)\right)=o_p(1)$.

Term: $\mathcal{S}_1^{\mathrm{PI}}$

First consider $\mathcal{S}_1^{\mathrm{PI}}$:

$$\begin{aligned}
\mathbb{E}\left|\mathcal{S}_1^{\mathrm{PI}}\right|^2 &= \mathbb{E}\left|\frac{n}{J^{d_z}T}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i_1\neq i_2}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{i_1t})\,\widehat{\mathbb{I}}_{jt}(z_{i_2t})\,\varepsilon_{i_1t}\varepsilon_{i_2t}\right|^2\\
&= \left(\frac{n}{J^{d_z}T}\right)^2\sum\nolimits_{t_1,t_2} n_{t_1}^{-2}n_{t_2}^{-2}\sum\nolimits_{\substack{i_1\neq i_2,\\ i_3\neq i_4}}\sum\nolimits_{j_1,j_2}\mathbb{E}\Big[\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\widehat{\mathbb{I}}_{j_2t_2}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-2}\times\\
&\qquad \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_1t_1}(z_{i_2t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_3t_2})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_4t_2})\,\varepsilon_{i_1t_1}\varepsilon_{i_2t_1}\varepsilon_{i_3t_2}\varepsilon_{i_4t_2}\Big].
\end{aligned}$$

The expectation is nonzero only when $(t_1=t_2)$ and either $(i_1=i_3),(i_2=i_4)$ or $(i_1=i_4),(i_2=i_3)$. This yields

$$\begin{aligned}
\mathbb{E}\left|\mathcal{S}_1^{\mathrm{PI}}\right|^2 &\leq C\left(\frac{n}{J^{d_z}T}\right)^2\sum\nolimits_t n_t^{-4}\sum\nolimits_{i_1,i_2}\sum\nolimits_{j_1}\mathbb{E}\left[\mathbf{1}_{j_1t}\widehat{\mathbb{I}}_{j_1t}(z)\,\hat{q}_{j_1t}^{-4}\widehat{\mathbb{I}}_{j_1t}(z_{i_1t})\,\widehat{\mathbb{I}}_{j_1t}(z_{i_2t})\,\varepsilon_{i_1t}^2\varepsilon_{i_2t}^2\right]\\
&\leq C\left(\frac{n}{J^{d_z}T}\right)^2\sum\nolimits_t n_t^{-4}\sum\nolimits_{i_1,i_2}\sum\nolimits_{j_1}\mathbb{E}\left[\mathbf{1}_{j_1t}\widehat{\mathbb{I}}_{j_1t}(z)\,\hat{q}_{j_1t}^{-4}\widehat{\mathbb{I}}_{j_1t}(z_{i_1t})\,\widehat{\mathbb{I}}_{j_1t}(z_{i_2t})\right]\\
&= C\left(\frac{n}{J^{d_z}T}\right)^2\sum\nolimits_t n_t^{-2}\sum\nolimits_{j_1}\mathbb{E}\left[\mathbf{1}_{j_1t}\widehat{\mathbb{I}}_{j_1t}(z)\,\hat{q}_{j_1t}^{-2}\right]\\
&\leq CJ^{2d_z}\left(\frac{n}{J^{d_z}T}\right)^2Tn^{-2}\\
&= CT^{-1},
\end{aligned}$$

so that $\mathcal{S}_1^{\mathrm{PI}}=O_p\left(T^{-1/2}\right)$ by Markov's inequality.

Term: $\mathcal{S}_2^{\mathrm{PI}}$

This term is

$$\begin{aligned}
&\mathcal{S}_2^{\mathrm{PI}}\\
=\;& \frac{2n}{J^{d_z}T}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\,\varepsilon_{it}\left(\hat{\varepsilon}_{it}-\varepsilon_{it}\right)\\
=\;& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1,t_2} n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1,i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\,\varepsilon_{i_1t_1}\varepsilon_{i_2t_2}\\
&-\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1,t_2} n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1,i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\times\\
&\qquad \varepsilon_{i_1t_1}\left(\mu(z_{i_2t_2})-\mu(z_{i_1t_1})\right)\\
&+\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1,t_2} n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1,i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\times
\end{aligned}$$

$$\varepsilon_{i_1t_1}\left[x'_{i_2t_2}\left(\hat{\beta}_{t_2}-\beta_{t_2}\right)-x'_{i_1t_1}\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\right]$$
$$-\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1,t_2}n_{t_1}^{-2}\sum\nolimits_{i_1}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\left(\mathbf{1}_{j_2t_2}-1\right)\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\varepsilon_{i_1t_1}\mu(z_{i_1t_1})$$
$$= \mathcal{S}_{21}^{\mathrm{PI}}+\mathcal{S}_{22}^{\mathrm{PI}}+\mathcal{S}_{23}^{\mathrm{PI}}+\mathcal{S}_{24}^{\mathrm{PI}}.$$

Term: $\mathcal{S}_{21}^{\mathrm{PI}}$ First consider $\mathcal{S}_{21}^{\mathrm{PI}}$ which can be decomposed as

$$\begin{aligned}
&\mathcal{S}_{21}^{\mathrm{PI}}\\
=\;& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1,t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1,i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\varepsilon_{i_1t_1}\varepsilon_{i_2t_2}\\
=\;& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1=t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1=i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\varepsilon_{i_1t_1}\varepsilon_{i_2t_2}\\
& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1=t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1\neq i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\varepsilon_{i_1t_1}\varepsilon_{i_2t_2}\\
& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1\neq t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1=i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\varepsilon_{i_1t_1}\varepsilon_{i_2t_2}\\
& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1\neq t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1\neq i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\varepsilon_{i_1t_1}\varepsilon_{i_2t_2}\\
=\;& \mathcal{S}_{211}^{\mathrm{PI}}+\mathcal{S}_{212}^{\mathrm{PI}}+\mathcal{S}_{213}^{\mathrm{PI}}+\mathcal{S}_{214}^{\mathrm{PI}}.
\end{aligned}$$

The first term is $\mathcal{S}_{211}^{\mathrm{PI}}$ which satisfies

$$\begin{aligned}
&\mathbb{E}\left|\mathcal{S}_{211}^{\mathrm{PI}}\right|\\
=\;& \mathbb{E}\left|\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1=t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1=i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\varepsilon_{i_1t_1}\varepsilon_{i_2t_2}\right|\\
=\;& \frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1}n_{t_1}^{-3}\sum\nolimits_{i_1}\sum\nolimits_{j_1}\mathbb{E}\left[\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-3}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\varepsilon_{i_1t_1}^2\right]\\
\leq\;& C\frac{n}{J^{d_z}T^2}\sum\nolimits_{t_1}n_{t_1}^{-2}\sum\nolimits_{j_1}\mathbb{E}\left[\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\right]\\
\leq\;& C\frac{J^{d_z}}{nT},
\end{aligned}$$

so that $\mathcal{S}_{211}^{\mathrm{PI}}=o_p(1)$ by Markov's inequality and our rate assumptions. Following similar steps, we can show that $\mathcal{S}_{212}^{\mathrm{PI}}=O_p\left(J^{d_z}n^{-1}T^{-3/2}\right)$, $\mathcal{S}_{213}^{\mathrm{PI}}=O_p\left(J^{3d_z/2}n^{-3/2}T^{-1}\right)$, and $\mathcal{S}_{214}^{\mathrm{PI}}=O_p\left(J^{d_z}n^{-1}T^{-1}\right)$ which are all $o_p(1)$ under our rate assumptions.

Term: $\mathcal{S}_{22}^{\mathrm{PI}}$ Next consider

$$\begin{aligned}
&\mathcal{S}_{22}^{\mathrm{PI}}\\
=\;& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1,t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1,i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\times\\
&\qquad \varepsilon_{i_1t_1}\left(\mu(z_{i_2t_2})-\mu(z_{i_1t_1})\right)\\
=\;& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1=t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1,i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\times\\
&\qquad \varepsilon_{i_1t_1}\left(\mu(z_{i_2t_2})-\mu(z_{i_1t_1})\right)\\
& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1\neq t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1=i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\times\\
&\qquad \varepsilon_{i_1t_1}\left(\mu(z_{i_2t_2})-\mu(z_{i_1t_1})\right)\\
& -\frac{2n}{J^{d_z}T^2}\sum\nolimits_{t_1\neq t_2}n_{t_1}^{-2}n_{t_2}^{-1}\sum\nolimits_{i_1\neq i_2}\sum\nolimits_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\times
\end{aligned}$$

$$
\begin{aligned}
& \varepsilon_{i_1 t_1}\left(\mu\left(z_{i_2 t_2}\right)-\mu\left(z_{i_1 t_1}\right)\right) \\
= \ & \mathcal{S}_{221}^{\mathrm{PI}}+\mathcal{S}_{222}^{\mathrm{PI}}+\mathcal{S}_{223}^{\mathrm{PI}}.
\end{aligned}
$$

Now consider,

$$
\begin{aligned}
& \mathbb{E}\left|\mathcal{S}_{221}^{\mathrm{PI}}\right| \\
\leq \ & \frac{2n}{J^{d_z}T^2}\sum_{t_1=t_2} n_{t_1}^{-2}n_{t_2}^{-1}\sum_{i_1,i_2}\sum_{j_1,j_2}\mathbb{E}\Big[\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\times \\
& \left|\varepsilon_{i_1t_1}\right|\left|\mu\left(z_{i_2t_2}\right)-\mu\left(z_{i_1t_1}\right)\right|\Big] \\
\leq \ & C\frac{n}{J^{d_z}T^2}\sum_{t_1=t_2} n_{t_1}^{-2}\sum_{i_1}\sum_{j_1,j_2}\mathbb{E}\left[\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\right] \\
\leq \ & C\frac{n}{J^{d_z}T^2}\sum_{t_1=t_2} n_{t_1}^{-1}\sum_{j_1}\mathbb{E}\left[\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-1}\right] \\
\leq \ & C\frac{1}{T}.
\end{aligned}
$$

Thus, $\mathcal{S}_{221}^{\mathrm{PI}}=o_p(1)$ by Markov's inequality. Following similar steps it can be shown that $\mathcal{S}_{222}^{\mathrm{PI}}=O_p\left(J^{d_z}n^{-1}T^{-1/2}\right)$ and $\mathcal{S}_{223}^{\mathrm{PI}}=O_p\left(J^{-d_z/2}n^{-1/2}T^{-1/2}\right)$ which are $o_p(1)$ under our rate assumptions.

Term: $\mathcal{S}_{23}^{\mathrm{PI}}$ Next consider $\mathcal{S}_{23}^{\mathrm{PI}}$

$$
\begin{aligned}
\mathcal{S}_{23}^{\mathrm{PI}} = \ & \frac{2n}{J^{d_z}T^2}\sum_{t_1,t_2} n_{t_1}^{-2}n_{t_2}^{-1}\sum_{i_1,i_2}\sum_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\times \\
& \varepsilon_{i_1t_1}\left[x_{i_2t_2}'\left(\hat{\beta}_{t_2}-\beta_{t_2}\right)-x_{i_1t_1}'\left(\hat{\beta}_{t_1}-\beta_{t_1}\right)\right] \\
= \ & \mathcal{S}_{231}^{\mathrm{PI}}+\mathcal{S}_{232}^{\mathrm{PI}}.
\end{aligned}
$$

First consider $\mathcal{S}_{231}^{\mathrm{PI}}$:

$$
\begin{aligned}
\mathcal{S}_{231}^{\mathrm{PI}} = \ & -\frac{2n}{J^{d_z}T^2}\sum_{t_1,t_2} n_{t_1}^{-2}n_{t_2}^{-1}\sum_{i_1,i_2}\sum_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1} \\
& \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\,\varepsilon_{i_1t_1}x_{i_2t_2}'\left(\hat{\beta}_{t_2}-\beta_{t_2}\right) \\
= \ & -\frac{2n}{J^{d_z}T^2}\sum_{t_2}\sum_{t_1} n_{t_1}^{-2}n_{t_2}^{-1}\sum_{i_1,i_2}\sum_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1} \\
& \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\,\varepsilon_{i_1t_1}x_{i_2t_2}'\left(\hat{\beta}_{t_2}-\beta_{t_2}\right)
\end{aligned}
$$

so that by the CS inequality

$$
\begin{aligned}
\left|\mathcal{S}_{231}^{\mathrm{PI}}\right|^2 \leq \ & \left(\frac{2n}{J^{d_z}T^2}\right)^2\times \\
& \sum_{t_2}\left(\sum_{t_1} n_{t_1}^{-2}n_{t_2}^{-1}\sum_{i_1,i_2}\sum_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\,\varepsilon_{i_1t_1}x_{i_2t_2}'\right) \\
& \times\left(\sum_{t_3} n_{t_3}^{-2}n_{t_2}^{-1}\sum_{i_3,i_4}\sum_{j_3,j_4}\mathbf{1}_{j_3t_3}\mathbf{1}_{j_4t_2}\widehat{\mathbb{I}}_{j_3t_3}(z)\,\hat{q}_{j_3t_3}^{-2}\hat{q}_{j_4t_2}^{-1}\widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_3t_3})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_4t_2})\,\varepsilon_{i_3t_3}x_{i_4t_2}'\right)' \\
& \sum_{t_2}\left(\hat{\beta}_{t_2}-\beta_{t_2}\right)'\left(\hat{\beta}_{t_2}-\beta_{t_2}\right)
\end{aligned}
$$

The third factor is

$$
\sum_{t_2}\left(\hat{\beta}_{t_2}-\beta_{t_2}\right)'\left(\hat{\beta}_{t_2}-\beta_{t_2}\right)=O_p\left(Tn^{-1}\right)+O_p\left(TJ^{-4}\right)
$$

The second factor has expectation

$$\begin{aligned}
&\mathbb{E}\sum_{t_2}\left(\sum_{t_1} n_{t_1}^{-2}n_{t_2}^{-1}\sum_{i_1,i_2}\sum_{j_1,j_2}\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\,\varepsilon_{i_1t_1}x_{i_2t_2}'\right)\\
&\times\left(\sum_{t_3} n_{t_3}^{-2}n_{t_2}^{-1}\sum_{i_3,i_4}\sum_{j_3,j_4}\mathbf{1}_{j_3t_3}\mathbf{1}_{j_4t_2}\widehat{\mathbb{I}}_{j_3t_3}(z)\,\hat{q}_{j_3t_3}^{-2}\hat{q}_{j_4t_2}^{-1}\widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_3t_3})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_4t_2})\,\varepsilon_{i_3t_3}x_{i_4t_2}'\right)'\\
=\;&\sum_{t_1,\ldots,t_3} n_{t_1}^{-2}n_{t_2}^{-2}n_{t_3}^{-2}\sum_{i_1,\ldots,i_4}\sum_{j_1,\ldots,j_4}\mathbb{E}\Big[\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\mathbf{1}_{j_3t_3}\mathbf{1}_{j_4t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\widehat{\mathbb{I}}_{j_3t_3}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\hat{q}_{j_3t_3}^{-2}\hat{q}_{j_4t_2}^{-1}\\
&\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\,\widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_3t_3})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_4t_2})\,\varepsilon_{i_1t_1}\varepsilon_{i_3t_3}x_{i_2t_2}'x_{i_4t_2}\Big].
\end{aligned}$$

The expectation is zero unless $(t_1=t_3, i_1=i_3)$ so we have

$$\begin{aligned}
&\sum_{t_1,\ldots,t_3} n_{t_1}^{-2}n_{t_2}^{-2}n_{t_3}^{-2}\sum_{i_1,\ldots,i_4}\sum_{j_1,\ldots,j_4}\mathbb{E}\Big[\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\mathbf{1}_{j_3t_3}\mathbf{1}_{j_4t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\widehat{\mathbb{I}}_{j_3t_3}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\hat{q}_{j_3t_3}^{-2}\hat{q}_{j_4t_2}^{-1}\\
&\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\,\widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_3t_3})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_4t_2})\,\varepsilon_{i_1t_1}\varepsilon_{i_3t_3}x_{i_2t_2}'x_{i_4t_2}\Big]\\
=\;&\sum_{t_1,t_2} n_{t_1}^{-4}n_{t_2}^{-2}\sum_{i_1,i_2,i_4}\sum_{j_1,\ldots,j_4}\mathbb{E}\Big[\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\mathbf{1}_{j_3t_3}\mathbf{1}_{j_4t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\widehat{\mathbb{I}}_{j_3t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-1}\hat{q}_{j_3t_1}^{-2}\hat{q}_{j_4t_2}^{-1}\\
&\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\,\widehat{\mathbb{I}}_{j_3t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_4t_2})\,\varepsilon_{i_1t_1}^2x_{i_2t_2}'x_{i_4t_2}\Big]\\
\leq\;& C\sum_{t_1,t_2} n_{t_1}^{-4}n_{t_2}^{-1}\sum_{i_1,i_2}\sum_{j_1,j_2,j_4}\mathbb{E}\Big[\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\mathbf{1}_{j_4t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-4}\hat{q}_{j_2t_2}^{-1}\hat{q}_{j_4t_2}^{-1}\\
&\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\,\widehat{\mathbb{I}}_{j_4t_2}(z_{i_1t_1})\left(n_{t_2}^{-1}\sum_{i_4}\widehat{\mathbb{I}}_{j_4t_2}(z_{i_4t_2})\right)\Big]\\
\leq\;& C\sum_{t_1,t_2} n_{t_1}^{-4}\sum_{i_1}\sum_{j_1}\mathbb{E}\left[\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-4}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\left(\sum_{j_2}\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\right)\right]\\
\leq\;& CT^2n^{-3}J^{3d_z}.
\end{aligned}$$

Thus,

$$\left|\mathcal{S}_{231}^{\mathrm{PI}}\right|^2\leq O\left(\left(\frac{n}{J^{d_z}T^2}\right)^2\right)\times O_p\left(T^2n^{-3}J^{3d_z}\right)\times\left[O_p\left(Tn^{-1}\right)+O_p\left(TJ^{-4}\right)\right],$$

by Markov's inequality and $\mathcal{S}_{231}^{\mathrm{PI}}=o_p(1)$ by our rate assumptions. By similar steps we can show that $\left|\mathcal{S}_{232}^{\mathrm{PI}}\right|^2=O\left(\left(\frac{n}{J^{d_z}T^2}\right)^2\right)O_p\left(J^{2d_z}n^{-2}T^3\right)\left[O_p\left(Tn^{-1}\right)+O_p\left(TJ^{-4}\right)\right]$ and so $\mathcal{S}_{232}^{\mathrm{PI}}=o_p(1)$ under our rate assumptions.

Term: $\mathcal{S}_{24}^{\mathrm{PI}}$ Finally, consider $\mathcal{S}_{24}^{\mathrm{PI}}$. This term satisfies,

$$\begin{aligned}
\left|\mathcal{S}_{24}^{\mathrm{PI}}\right|\;\leq\;&\frac{2n}{J^{d_z}T^2}\sum_{t_1,t_2} n_{t_1}^{-2}\sum_{i_1}\sum_{j_1,j_2}\mathbf{1}_{j_1t_1}\left|\mathbf{1}_{j_2t_2}-1\right|\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\left|\varepsilon_{i_1t_1}\right|\left|\mu(z_{i_1t_1})\right|\\
\leq\;&\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\left|\mathbf{1}_{jt}-1\right|\times C\frac{n}{J^{d_z}T^2}\sum_{t_1,t_2} n_{t_1}^{-2}\sum_{i_1}\sum_{j_1,j_2}\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\left|\varepsilon_{i_1t_1}\right|.
\end{aligned}$$

The second factor has expectation

$$\begin{aligned}
&C\frac{n}{J^{d_z}T^2}\sum_{t_1,t_2} n_{t_1}^{-2}\sum_{i_1}\sum_{j_1,j_2}\mathbb{E}\left[\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-2}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\,\widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\left|\varepsilon_{i_1t_1}\right|\right]\\
\leq\;& C\frac{n}{J^{d_z}T^2}\sum_{t_1,t_2} n_{t_1}^{-1}\sum_{j_1}\mathbb{E}\left[\mathbf{1}_{j_1t_1}\widehat{\mathbb{I}}_{j_1t_1}(z)\,\hat{q}_{j_1t_1}^{-1}\right]\\
\leq\;& C.
\end{aligned}$$

Thus this term is $o_p(1)$ by Lemma 2 .

Term: $\mathcal{S}_{3}^{\mathrm{PI}}$

We have that

$$\begin{aligned}
\mathcal{S}_3^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z)\, \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it}) \left(\hat{\varepsilon}_{it} - \varepsilon_{it}\right)^2 \\
&= \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \sum\nolimits_{i_2} \hat{q}_{j_2t_2}^{-1} \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\, \varepsilon_{i_2t_2}\right]^2 \\
&\quad + \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2t_2} \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \left(\mu(z_{i_2t_2}) - \mu(z_{i_1t_1})\right)\right]^2 \\
&\quad + \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2t_2} \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \left[x_{i_2t_2}' \left(\hat{\beta}_{t_2} - \beta_{t_2}\right) - x_{i_1t_1}' \left(\hat{\beta}_{t_1} - \beta_{t_1}\right)\right]\right]^2 \\
&\quad + \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} \sum\nolimits_{j_2} \left(\mathbf{1}_{j_2t_2} - 1\right) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\, \mu(z_{i_1t_1})\right]^2 \\
&= \mathcal{S}_{31}^{\mathrm{PI}} + \mathcal{S}_{32}^{\mathrm{PI}} + \mathcal{S}_{33}^{\mathrm{PI}} + \mathcal{S}_{34}^{\mathrm{PI}}.
\end{aligned}$$

Term: $\mathcal{S}_{31}^{\mathrm{PI}}$ Now consider $\mathcal{S}_{31}^{\mathrm{PI}}$. It is

$$\begin{aligned}
\mathcal{S}_{31}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1,t_2,t_3} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1,j_2,j_3} \mathbf{1}_{j_1t_1} \mathbf{1}_{j_2t_2} \mathbf{1}_{j_3t_3} \times \\
&\quad \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \hat{q}_{j_2t_2}^{-1} \hat{q}_{j_3t_3}^{-1} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\, \varepsilon_{i_2t_2} \varepsilon_{i_3t_3} \\
&= \sum\nolimits_{\ell=1}^{5} \mathcal{S}_{31\ell}^{\mathrm{PI}},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}_{311}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1=t_2=t_3} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1,j_2,j_3} \mathbf{1}_{j_1t_1} \mathbf{1}_{j_2t_2} \mathbf{1}_{j_3t_3} \times \\
&\quad \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \hat{q}_{j_2t_2}^{-1} \hat{q}_{j_3t_3}^{-1} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\, \varepsilon_{i_2t_2} \varepsilon_{i_3t_3} \\
\mathcal{S}_{312}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1\neq t_2=t_3} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1,j_2,j_3} \mathbf{1}_{j_1t_1} \mathbf{1}_{j_2t_2} \mathbf{1}_{j_3t_3} \times \\
&\quad \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \hat{q}_{j_2t_2}^{-1} \hat{q}_{j_3t_3}^{-1} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\, \varepsilon_{i_2t_2} \varepsilon_{i_3t_3} \\
\mathcal{S}_{313}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1=t_2\neq t_3} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1,j_2,j_3} \mathbf{1}_{j_1t_1} \mathbf{1}_{j_2t_2} \mathbf{1}_{j_3t_3} \times \\
&\quad \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \hat{q}_{j_2t_2}^{-1} \hat{q}_{j_3t_3}^{-1} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\, \varepsilon_{i_2t_2} \varepsilon_{i_3t_3} \\
\mathcal{S}_{314}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1=t_3\neq t_2} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1,j_2,j_3} \mathbf{1}_{j_1t_1} \mathbf{1}_{j_2t_2} \mathbf{1}_{j_3t_3} \times \\
&\quad \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \hat{q}_{j_2t_2}^{-1} \hat{q}_{j_3t_3}^{-1} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\, \varepsilon_{i_2t_2} \varepsilon_{i_3t_3} \\
\mathcal{S}_{315}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1\neq t_2\neq t_3} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1,j_2,j_3} \mathbf{1}_{j_1t_1} \mathbf{1}_{j_2t_2} \mathbf{1}_{j_3t_3} \times \\
&\quad \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \hat{q}_{j_2t_2}^{-1} \hat{q}_{j_3t_3}^{-1} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\, \varepsilon_{i_2t_2} \varepsilon_{i_3t_3}
\end{aligned}$$

Term: $\mathcal{S}_{311}^{\mathrm{PI}}$ First consider $\mathcal{S}_{311}^{\mathrm{PI}}$:

$$\begin{aligned}
\mathcal{S}_{311}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1=t_2=t_3} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1,j_2,j_3} \mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\mathbf{1}_{j_3t_3} \times \\
&\quad \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \hat{q}_{j_2t_2}^{-1} \hat{q}_{j_3t_3}^{-1} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \widehat{\mathbb{I}}_{j_3t_3}(z_{i_3t_3})\, \varepsilon_{i_2t_2}\varepsilon_{i_3t_3} \\
&= \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1,j_2,j_3} \mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_1}\mathbf{1}_{j_3t_1} \times \\
&\quad \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \hat{q}_{j_2t_1}^{-1} \hat{q}_{j_3t_1}^{-1} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_3t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_1}(z_{i_2t_1}) \widehat{\mathbb{I}}_{j_3t_1}(z_{i_3t_1})\, \varepsilon_{i_2t_1}\varepsilon_{i_3t_1} \\
&= \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1} \mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-4} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_1t_1}(z_{i_2t_1}) \widehat{\mathbb{I}}_{j_1t_1}(z_{i_3t_1})\, \varepsilon_{i_2t_1}\varepsilon_{i_3t_1}
\end{aligned}$$

which satisfies

$$\begin{aligned}
&\mathbb{E}\left|\mathcal{S}_{311}^{\mathrm{PI}}\right| \\
\leq\ & \left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1} n_{t_1}^{-2} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \sum\nolimits_{j_1} \mathbb{E}\left[\mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-4} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_1t_1}(z_{i_2t_1}) \widehat{\mathbb{I}}_{j_1t_1}(z_{i_3t_1}) \left|\varepsilon_{i_2t_1}\varepsilon_{i_3t_1}\right|\right] \\
\leq\ & C\left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1} n_{t_1}^{-1} \sum\nolimits_{j_1} \mathbb{E}\left[\mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-4} \left(n_{t_1}^{-1} n_{t_2}^{-1} n_{t_3}^{-1} \sum\nolimits_{i_1,i_2,i_3} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_1t_1}(z_{i_2t_1}) \widehat{\mathbb{I}}_{j_1t_1}(z_{i_3t_1})\right)\right] \\
=\ & C\left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1} n_{t_1}^{-1} \sum\nolimits_{j_1} \mathbb{E}\left[\mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-1}\right] \\
\leq\ & C\left(\frac{nT}{J^{d_z}}\right) T^{-4} J^{d_z} \sum\nolimits_{t_1} n_{t_1}^{-1} \sum\nolimits_{j_1} \mathbb{E}\left[\widehat{\mathbb{I}}_{j_1t_1}(z)\right] \\
\leq\ & CT^{-2},
\end{aligned}$$

so that $\mathcal{S}_{311}^{\mathrm{PI}} = O_p\left(T^{-2}\right) = o_p(1)$ by Markov's inequality. By similar steps we can show that $\mathcal{S}_{312}^{\mathrm{PI}} = O_p\left(T^{-1}\right) =$, $\mathcal{S}_{313}^{\mathrm{PI}} = O_p\left(T^{-1}\right)$, $\mathcal{S}_{314}^{\mathrm{PI}} = O_p\left(T^{-1}\right)$, and $\mathcal{S}_{315}^{\mathrm{PI}} = O_p\left(J^{d_z} n^{-1}\right)$ which are $o_p(1)$ under our rate assumptions.

Term: $\mathcal{S}_{32}^{\mathrm{PI}}$ Now consider $\mathcal{S}_{32}^{\mathrm{PI}}$:

$$\begin{aligned}
\mathcal{S}_{32}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2t_2} \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2}) \left(\mu(z_{i_2t_2}) - \mu(z_{i_1t_1})\right)\right]^2 \\
&\leq C\left[\max_{1\leq t\leq T} \max_{1\leq j\leq J_t^{d_z}} \sup_z \left|\widehat{\mathbb{I}}_{jt}(z)\mu(z) - \widehat{\mathbb{I}}_{jt}(z)\gamma_{jt}^0\right|^2\right] \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \\
&\quad \times \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2t_2} \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\right]^2
\end{aligned}$$

The first factor is $O_p\left(J^{-2}\right)$ by Lemma 1. The second factor is

$$\begin{aligned}
&\leq \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2t_2} \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1}) \widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\right]^2 \\
&\leq \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1t_1} \widehat{\mathbb{I}}_{j_1t_1}(z)\, \hat{q}_{j_1t_1}^{-2} \widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} \sum\nolimits_{j_2} \widehat{\mathbb{I}}_{j_2t_2}(z_{i_1t_1})\right]^2
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \left(n_{t_1}^{-1} \sum\nolimits_{i_1} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1})\right) \\
&\leq C.
\end{aligned}
$$

Thus $\mathcal{S}_{32}^{\mathrm{PI}} = O_p\left(J^{-2}\right) = o\left(1\right)$.

Term: $\mathcal{S}_{33}^{\mathrm{PI}}$ Now consider $\mathcal{S}_{33}^{\mathrm{PI}}$

$$
\begin{aligned}
\mathcal{S}_{33}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2 t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2 t_2} \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_1 t_1}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_2 t_2}) \left[x_{i_2 t_2}' \left(\hat{\beta}_{t_2} - \beta_{t_2}\right) - x_{i_1 t_1}' \left(\hat{\beta}_{t_1} - \beta_{t_1}\right)\right]\right]^2 \\
&\leq 2\left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2 t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2 t_2} \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_1 t_1}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_2 t_2})\, x_{i_2 t_2}' \left(\hat{\beta}_{t_2} - \beta_{t_2}\right)\right]^2 \\
&\quad + 2\left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2 t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2 t_2} \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_1 t_1}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_2 t_2})\, x_{i_1 t_1}' \left(\hat{\beta}_{t_1} - \beta_{t_1}\right)\right]^2 \\
&= \mathcal{S}_{331}^{\mathrm{PI}} + \mathcal{S}_{332}^{\mathrm{PI}}.
\end{aligned}
$$

First consider $\mathcal{S}_{331}^{\mathrm{PI}}$. By the CS inequality we have,

$$
\begin{aligned}
\mathcal{S}_{331}^{\mathrm{PI}} &= 2\left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}) \times \\
&\quad \left[T^{-1} \sum\nolimits_{t_2} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2 t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2 t_2} \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_1 t_1}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_2 t_2})\, x_{i_2 t_2}' \left(\hat{\beta}_{t_2} - \beta_{t_2}\right)\right]^2 \\
&\leq 2\left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}) \times \\
&\quad \sum\nolimits_{t_2} \left(T^{-1} n_{t_2}^{-1} \sum\nolimits_{j_2} \hat{q}_{j_2 t_2}^{-1} \sum\nolimits_{i_2} \mathbf{1}_{j_2 t_2} \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_1 t_1}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_2 t_2})\, x_{i_2 t_2}'\right)^2 \times \\
&\quad \left[\sum\nolimits_{t_3} \left(\hat{\beta}_{t_3} - \beta_{t_3}\right)' \left(\hat{\beta}_{t_3} - \beta_{t_3}\right)\right]^2.
\end{aligned}
$$

The last factor is $O_p\left(Tn^{-1}\right) + \ O_p\left(TJ^{-4}\right)$ The first and second factor are then bounded by

$$
\begin{aligned}
& 2\left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1, t_2} n_{t_1}^{-2} n_{t_2}^{-2} \sum\nolimits_{i_1, i_2, i_3} \sum\nolimits_{j_1, j_2} \mathbb{E}\Big[\mathbf{1}_{q, j_1 t_1} \mathbf{1}_{q, j_2 t_2} \mathbf{1}_{q, j_3 t_3} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2}^{-2} \times \\
& \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_1 t_1}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_2 t_2}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_3 t_2})\, \mathbb{E}\left[x_{i_2 t_2}' x_{i_3 t_2} \middle| z_{t_1}, z_{t_2}, z_{t_3}, \mathcal{F}_{t_2}\right]\Big] \\
\leq\ & C\left(\frac{nT}{J^{d_z}}\right) T^{-4} \sum\nolimits_{t_1, t_2} n_{t_1}^{-2} \sum\nolimits_{i_1,} \sum\nolimits_{j_1, j_2} \mathbb{E}\Big[\mathbf{1}_{q, j_1 t_1} \mathbf{1}_{q, j_2 t_2} \mathbf{1}_{q, j_3 t_3} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \hat{q}_{j_2 t_2}^{-2} \times \\
& \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_1 t_1}) \left(n_{t_2}^{-2} \sum\nolimits_{i_2, i_3} \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_2 t_2}) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_3 t_2})\right)\Big] \\
\leq\ & C\left(\frac{nT}{J^{d_z}}\right) T^{-4} J^{d_z} \sum\nolimits_{t_1, t_2} n_{t_1}^{-1} \sum\nolimits_{j_1} \mathbb{E}\left[\widehat{\mathbb{I}}_{j_1 t_1}(z)\right] \\
\leq\ & CT^{-1}.
\end{aligned}
$$

Thus, $\mathcal{S}_{331}^{\mathrm{PI}} = O_p\left(n^{-1}\right) + \ O_p\left(J^{-4}\right)$. By similar steps we can show that $\mathcal{S}_{332}^{\mathrm{PI}} = O_p\left(n^{-1}\right) + O_p\left(T^{-1}J^{-2}\right)$ which is $o_p\left(1\right)$ under our rate assumptions.

Term: $\mathcal{S}_{34}^{\mathrm{PI}}$ Now consider $\mathcal{S}_{34}^{\mathrm{PI}}$:

$$\begin{aligned}\mathcal{S}_{34}^{\mathrm{PI}} &= \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}) \times \\ &\quad \left[T^{-1} \sum\nolimits_{t_2} \sum\nolimits_{j_2} (\mathbf{1}_{j_2 t_2} - 1) \widehat{\mathbb{I}}_{j_2 t_2}(z_{i_1 t_1})\, \mu(z_{i_1 t_1})\right]^2,\end{aligned}$$

which satisfies

$$\left|\mathcal{S}_{34}^{\mathrm{PI}}\right| \leq \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t^{d_z}} |\mathbf{1}_{jt} - 1| \times C \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}).$$

The second factor is

$$\begin{aligned}& \left(\frac{nT}{J^{d_z}}\right) T^{-2} \sum\nolimits_{t_1} n_{t_1}^{-2} \sum\nolimits_{i_1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1}) \\ =\;& \frac{n}{J^{d_z} T} \sum\nolimits_{t_1} n_{t_1}^{-1} \sum\nolimits_{j_1} \mathbf{1}_{j_1 t_1} \widehat{\mathbb{I}}_{j_1 t_1}(z)\, \hat{q}_{j_1 t_1}^{-2} \left(n_{t_1}^{-1} \sum\nolimits_{i_1} \widehat{\mathbb{I}}_{j_1 t_1}(z_{i_1 t_1})\right) \\ \leq\;& C \frac{n}{T} \sum\nolimits_{t_1} n_{t_1}^{-1} \sum\nolimits_{j_1} \widehat{\mathbb{I}}_{j_1 t_1}(z) \\ \leq\;& C.\end{aligned}$$

Thus, $\mathcal{S}_{34}^{\mathrm{PI}} = o_p(1)$ by Lemma 2.

B.7 Proof of Theorem 3

As discussed in the main text we will work with a modified version of $\mathcal{L}_2$ and $\mathcal{L}_3$ where we assume that the conditional quantiles are known. We start with

$$\begin{aligned}\mathcal{L}_1 &= T^{-1} \sum\nolimits_{t=1}^{T} n_t^{-1} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \mathbb{I}_{jt}(z)\, \tilde{q}_{jt}^{-1} \mathbb{I}_{jt}(z_{it}) \left(\mu(z_{it}) - \mu(z)\right), \\ \mathcal{L}_{21} &= T^{-1} \sum\nolimits_{t=1}^{T} n_t^{-1} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \mathbb{I}_{jt}(z)\, q_{jt}^{-1} \mathbb{I}_{jt}(z_{it})\, \varepsilon_{it} \\ \mathcal{L}_{22} &= -T^{-1} \sum\nolimits_{t=1}^{T} n_t^{-1} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \mathbb{I}_{jt}(z)\, q_{jt}^{-2} (\tilde{q}_{jt} - q_{jt})\, \mathbb{I}_{jt}(z_{it})\, \varepsilon_{it} \\ \mathcal{L}_{23} &= T^{-1} \sum\nolimits_{t=1}^{T} n_t^{-1} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \mathbb{I}_{jt}(z)\, \tilde{q}_{jt}^{-1} q_{jt}^{-2} (\tilde{q}_{jt} - q_{jt})^2\, \mathbb{I}_{jt}(z_{it})\, \varepsilon_{it} \\ \mathcal{L}_3 &= -T^{-1} \sum\nolimits_{t=1}^{T} n_t^{-1} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \mathbb{I}_{jt}(z)\, \tilde{q}_{jt}^{-1} \mathbb{I}_{jt}(z_{it})\, x_{it}' \left(\hat{\beta}_t - \beta_t\right), \\ \mathcal{L}_4 &= T^{-1} \sum\nolimits_{t=1}^{T} \sum\nolimits_{j=1}^{J_t^{d_z}} (\mathbf{1}_{jt} - 1)\, \mathbb{I}_{jt}(z)\, \mu(z),\end{aligned}$$

and recall that $\tilde{q}_{jt} = n_t^{-1} \sum_{i=1}^{n_t} \mathbb{I}_{jt}(z_{it})$ so that

$$\hat{\mu}(z) - \mu(z) = \mathcal{L}_1 + \mathcal{L}_{21} + \mathcal{L}_{22} + \mathcal{L}_{23} + \mathcal{L}_3 + \mathcal{L}_4.$$

Thus,

$$\mathbb{E}\left[\left|\hat{\mu}(z) - \mu(z)\right|^2 \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T\right] = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5 + o_p\left(J^{-2} + \frac{J^{2d_z}}{n^2 T}\right),$$

where

$$\begin{aligned}
\mathcal{M}_1 &= \mathcal{L}_1^2 \\
\mathcal{M}_2 &= \mathbb{E}\left[\left(\mathcal{L}_{21}+\mathcal{L}_{22}+\mathcal{L}_{23}\right)^2 \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right] \\
\mathcal{M}_3 &= \mathbb{E}\left[\mathcal{L}_3^2 \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right] \\
\mathcal{M}_4 &= 2\mathcal{L}_1 \mathbb{E}\left[\mathcal{L}_3 \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right] \\
\mathcal{M}_5 &= 2\mathbb{E}\left[\left(\mathcal{L}_{21}+\mathcal{L}_{22}+\mathcal{L}_{23}\right)\mathcal{L}_3 \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right]
\end{aligned}$$

since $\mathbb{E}\left[\mathcal{L}_1\left(\mathcal{L}_{21}+\mathcal{L}_{22}+\mathcal{L}_{23}\right) \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right]=0$ and by Lemmas 2 and 3 all terms involving $\mathcal{L}_4$ are of smaller order.

B.7.1 Term: $\mathcal{M}_1$

First, consider, $\mathcal{L}_1^2$. For this term we work with estimated quantiles. We have,

$$\begin{aligned}
\mathcal{L}_1 &= T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left(\mu(z_{it})-\mu(z)\right) \\
&= \frac{\partial\mu(z)}{\partial z'}\cdot T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left(z_{it}-z\right) \\
&\quad + T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left(z_{it}-z\right)' \left.\frac{\partial\mu(z)}{\partial z\partial z'}\right|_{z=\tilde{z}}\left(z_{it}-z\right) \\
&= \mathcal{L}_{11}+\mathcal{L}_{12},
\end{aligned}$$

where $\tilde{z}=\alpha z+(1-\alpha)z_{it}$, $\alpha\in(0,1)$ Thus, we need only show that $\mathcal{L}_{12}$ is $o_p\left(J^{-1}\right)$. We have,

$$\begin{aligned}
\left|\mathcal{L}_{12}\right| &\le T^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left|\left(z_{it}-z\right)' \left.\frac{\partial\mu(z)}{\partial z\partial z'}\right|_{z=\tilde{z}}\left(z_{it}-z\right)\right| \\
&\le CT^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\left\|z_{it}-z\right\|^2 \\
&\le CT^{-1}\sum_{t=1}^{T} n_t^{-1}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1}\widehat{\mathbb{I}}_{jt}(z_{it})\sum_{s=1}^{d_z}\left(\hat{b}_{j_s t,s}-\hat{b}_{(j_s-1)t,s}\right)^2 \\
&= CT^{-1}\sum_{t=1}^{T}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\sum_{s=1}^{d_z}\left(\hat{b}_{j_s t,s}-\hat{b}_{(j_s-1)t,s}\right)^2 \\
&\le \max_{1\le t\le T}\max_{1\le j\le J^{d_z}}\max_{1\le s\le d_z}\left|\hat{b}_{j_s t,s}-\hat{b}_{(j_s-1)t,s}\right|^2\times CT^{-1}\sum_{t=1}^{T}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z),
\end{aligned}$$

and so $\mathcal{L}_{12}=O_p\left(J^{-2}\right)$ by Lemma 1 and the result follows.

B.7.2 Term: $\mathcal{M}_2$

We have

$$\begin{aligned}
\mathcal{M}_{21} &= \mathbb{E}\left[\left|\mathcal{L}_{21}\right|^2 \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right] \\
\mathcal{M}_{22} &= \mathbb{E}\left[\left|\mathcal{L}_{22}\right|^2 \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right] \\
\mathcal{M}_{23} &= \mathbb{E}\left[\left|\mathcal{L}_{23}\right|^2 \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right] \\
\mathcal{M}_{24} &= 2\mathbb{E}\left[\mathcal{L}_{21}\mathcal{L}_{22} \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right] \\
\mathcal{M}_{25} &= 2\mathbb{E}\left[\mathcal{L}_{21}\mathcal{L}_{23} \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right] \\
\mathcal{M}_{26} &= 2\mathbb{E}\left[\mathcal{L}_{22}\mathcal{L}_{23} \middle| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1,\ldots,\mathcal{F}_T \right].
\end{aligned}$$

Squared Terms

First note that $\mathcal{M}_{21}$ is

$$\begin{aligned}\mathcal{M}_{21} &= T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\, q_{jt}^{-2}\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2 \\ &= \mathcal{M}_{211}+\mathcal{M}_{212}+o_p\left(J^{2d_z}n^{-2}T^{-1}\right),\end{aligned}$$

by Lemma 2 where

$$\begin{aligned}\mathcal{M}_{211} &= T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z)\, q_{jt}^{-2}\mathbb{E}\left[\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2\middle|\,\mathcal{F}_t\right] \\ \mathcal{M}_{212} &= T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z)\, q_{jt}^{-2}\left\{\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2-\mathbb{E}\left[\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2\middle|\,\mathcal{F}_t\right]\right\}.\end{aligned}$$

Note that $\mathcal{M}_{211}=O_p\left(J^{d_z}n^{-1}T^{-1}\right)$ as given in the proof of Theorem 1. Next, $\mathcal{M}_{212}$ is mean zero with variance,

$$\begin{aligned}\mathbb{E}\left[\left|\mathcal{M}_{212}\right|^2\right] &= T^{-4}\sum\nolimits_{t=1}^{T} n_t^{-4}\sum\nolimits_{i_1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbb{I}_{jt}(z)\, q_{jt}^{-4}\left(\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2-\mathbb{E}\left[\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2\middle|\,\mathcal{F}_t\right]\right)^2\right] \\ &\leq CT^{-3}n^{-3}J^{3d_z},\end{aligned}$$

where the first equality follows by Assumption 1. Thus, $\mathcal{M}_{212}=O_p\left(J^{3d_z/2}n^{-3/2}T^{-3/2}\right)$.

Next, we have

$$\begin{aligned}\mathcal{M}_{22} &= T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\, q_{jt}^{-4}\left(\tilde{q}_{jt}-q_{jt}\right)^2\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2 \\ &= T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-4}\sum\nolimits_{i_1,i_2,i_3}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\, q_{jt}^{-4}\left(\mathbb{I}_{jt}(z_{i_2t})-q_{jt}\right)\left(\mathbb{I}_{jt}(z_{i_3t})-q_{jt}\right)\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2 \\ &= \mathcal{M}_{221}+\mathcal{M}_{222}+\mathcal{M}_{223}+\mathcal{M}_{224},\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_{221} &= T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-4}\sum\nolimits_{i_1=i_2=i_3}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\, q_{jt}^{-4}\left(\mathbb{I}_{jt}(z_{i_2t})-q_{jt}\right)\left(\mathbb{I}_{jt}(z_{i_3t})-q_{jt}\right)\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2 \\ \mathcal{M}_{222} &= T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-4}\sum\nolimits_{i_1\neq i_2=i_3}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\, q_{jt}^{-4}\left(\mathbb{I}_{jt}(z_{i_2t})-q_{jt}\right)\left(\mathbb{I}_{jt}(z_{i_3t})-q_{jt}\right)\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2 \\ \mathcal{M}_{223} &= 2T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-4}\sum\nolimits_{i_1\neq i_2, i_1=i_3}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\, q_{jt}^{-4}\left(\mathbb{I}_{jt}(z_{i_2t})-q_{jt}\right)\left(\mathbb{I}_{jt}(z_{i_3t})-q_{jt}\right)\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2 \\ \mathcal{M}_{224} &= T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-4}\sum\nolimits_{i_1\neq i_2\neq i_3}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\, q_{jt}^{-4}\left(\mathbb{I}_{jt}(z_{i_2t})-q_{jt}\right)\left(\mathbb{I}_{jt}(z_{i_3t})-q_{jt}\right)\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2.\end{aligned}$$

However,

$$\mathbb{E}\left|\mathcal{M}_{221}\right| = T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-4}\sum\nolimits_{i_1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\mathbb{E}\left[q_{jt}^{-4}\left(1-q_{jt}\right)^2\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2\right]\leq CT^{-1}n^{-3}J^{3d},$$

so that $\mathcal{M}_{221}=O_p\left(T^{-1}n^{-3}J^{3d}\right)=o_p\left(J^{2d}n^{-2}T^{-1}\right)$ by Markov's inequality and Assumption 3. Next,

$$\begin{aligned}\mathcal{M}_{222} &= T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-4}\sum\nolimits_{i_1\neq i_2}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\, q_{jt}^{-4}\left(\mathbb{I}_{jt}(z_{i_2t})-q_{jt}\right)^2\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2 \\ &= \mathcal{M}_{2221}+\mathcal{M}_{2222}+o_p\left(\frac{J^{2d}}{n^2T}\right),\end{aligned}$$

by Lemma 2 where

$$\begin{aligned}
\mathcal{M}_{2221} &= T^{-2}\sum_{t=1}^{T} n_t^{-4}(n_t-1)\sum_{i_1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z)\, q_{jt}^{-3}\mathbb{E}\left[\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2\middle|\mathcal{F}_t\right]\\
\mathcal{M}_{2222} &= T^{-2}\sum_{t=1}^{T} n_t^{-4}\sum_{i_1\neq i_2}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z)\, q_{jt}^{-4}\times\\
&\quad \left\{(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})^2\,\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2-\mathbb{E}\left[(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})^2\,\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2\middle|\mathcal{F}_t\right]\right\}.
\end{aligned}$$

$\mathcal{M}_{2221}=O_p\left(J^{2d_z}n^{-2}T^{-1}\right)$. Next note that $\mathcal{M}_{2222}$ is mean zero with variance,

$$\begin{aligned}
\mathbb{E}\left[|\mathcal{M}_{2222}|^2\right] &\leq T^{-4}\sum_{t=1}^{T} n_t^{-8}\sum_{i_1\neq i_2, i_3\neq i_4}^{n_t}\sum_{j=1}^{J_t^{d_z}}\\
&\quad \mathbb{E}\left[\mathbb{I}_{jt}(z)\,q_{jt}^{-8}\left\{(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})^2\,\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2-\mathbb{E}\left[(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})^2\,\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2\middle|\mathcal{F}_t\right]\right\}\right.\\
&\quad \left.\left\{(\mathbb{I}_{jt}(z_{i_4t})-q_{jt})^2\,\mathbb{I}_{jt}(z_{i_3t})\,\sigma_{i_3t}^2-\mathbb{E}\left[(\mathbb{I}_{jt}(z_{i_4t})-q_{jt})^2\,\mathbb{I}_{jt}(z_{i_3t})\,\sigma_{i_3t}^2\middle|\mathcal{F}_t\right]\right\}\right].
\end{aligned}$$

There are six nonzero terms and by Markov's inequality and following similar steps as above we can show that $\mathcal{M}_{2222}=O_p\left(T^{-3/2}n^{-5/2}J^{5d_z/2}\right)=o_p\left(J^{2d_z}n^{-2}T^{-1}\right)$ by Assumption 3. Next,

$$\mathcal{M}_{223}=2T^{-2}\sum_{t=1}^{T} n_t^{-4}\sum_{i_1\neq i_2}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,q_{jt}^{-4}(1-q_{jt})(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})\,\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2$$

is mean zero with variance

$$\begin{aligned}
&\mathbb{E}\left[|\mathcal{M}_{223}|^2\right]\\
=\;& 4T^{-4}\sum_{t=1}^{T} n_t^{-8}\sum_{i_1\neq i_2, i_3\neq i_4}^{n_t}\sum_{j=1}^{J_t^{d_z}}\\
&\mathbb{E}\left[\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,q_{jt}^{-8}(1-q_{jt})^2(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})(\mathbb{I}_{jt}(z_{i_4t})-q_{jt})\,\mathbb{I}_{jt}(z_{i_1t})\,\mathbb{I}_{jt}(z_{i_3t})\,\sigma_{i_1t}^2\sigma_{i_3t}^2\right].
\end{aligned}$$

There are three nonzero terms and by Markov's inequality and following similar steps as above we can show that $\mathcal{M}_{223}=O_p\left(J^{5d_z/2}n^{-5/2}T^{-3/2}\right)=o_p\left(J^{2d_z}n^{-2}T^{-1}\right)$ by Assumption 3.

Finally, we have $\mathcal{M}_{224}$,

$$\mathcal{M}_{224}=T^{-2}\sum_{t=1}^{T} n_t^{-4}\sum_{\substack{i_1\neq i_2, i_2\neq i_3\\ i_1\neq i_3}}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,q_{jt}^{-4}(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})(\mathbb{I}_{jt}(z_{i_3t})-q_{jt})\,\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2,$$

which is mean zero with variance

$$\begin{aligned}
&\mathbb{E}\left[|\mathcal{M}_{224}|^2\right]\\
=\;& T^{-4}\sum_{t=1}^{T} n_t^{-8}\sum_{\substack{i_1\neq i_2, i_2\neq i_3, i_1\neq i_3\\ i_4\neq i_5, i_5\neq i_6, i_4\neq i_6}}^{n_t}\sum_{j=1}^{J_t^{d_z}}\\
&\mathbb{E}\left[\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,q_{jt}^{-8}(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})(\mathbb{I}_{jt}(z_{i_3t})-q_{jt})(\mathbb{I}_{jt}(z_{i_5t})-q_{jt})(\mathbb{I}_{jt}(z_{i_6t})-q_{jt})\,\mathbb{I}_{jt}(z_{i_1t})\,\mathbb{I}_{jt}(z_{i_4t})\,\sigma_{i_1t}^2\sigma_{i_4t}^2\right].
\end{aligned}$$

There are four nonzero terms and by Markov's inequality and following similar steps as above we can show that $\mathcal{M}_{224}=O_p\left(T^{-3/2}n^{-2}J^{2d_z}\right)=o_p\left(J^{2d_z}n^{-2}T^{-1}\right)$ by Markov's inequality and Assumption 3.

Next we have

$$\mathcal{M}_{23}=\mathbb{E}\left[|\mathcal{L}_{23}|^2\middle|\mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right]=T^{-2}\sum_{t=1}^{T} n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-2}q_{jt}^{-4}(\tilde{q}_{jt}-q_{jt})^4\,\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2,$$

and note that

$$
\begin{aligned}
& T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-2}q_{jt}^{-4}(\tilde{q}_{jt}-q_{jt})^4\,\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2 \\
\leq\ & CT^{-2}\sum\nolimits_{t=1}^{T} n_t^{-1}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}q_{jt}^{-4}(\tilde{q}_{jt}-q_{jt})^4 \\
\leq\ & CT^{-2}\cdot\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}} q_{jt}^{-4}\left|\hat{q}_{jt}-q_{jt}\right|^4\cdot J^{d_z}n^{-1},
\end{aligned}
$$

so that $\mathcal{M}_{23}=O_p\left(T^{-2}n^{-3}J^{3d_z}\log\left(J^{d_z}\vee T\right)^2\right)=o_p\left(J^{2d_z}n^{-2}T^{-1}\right)$.

Cross-Product Terms

The first cross-product term is:

$$
\begin{aligned}
\mathcal{M}_{24} &= 2\mathbb{E}\left[\mathcal{L}_{21}\mathcal{L}_{22}\right|\mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T] \\
&= -2T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,q_{jt}^{-3}(\tilde{q}_{jt}-q_{jt})\,\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2 \\
&= -2T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-3}\sum\nolimits_{i_1,i_2}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,q_{jt}^{-3}(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})\,\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2 \\
&= \mathcal{M}_{241}+\mathcal{M}_{242}+\mathcal{M}_{243}+o_p\left(\frac{J^{2d_z}}{n^2T}\right)
\end{aligned}
$$

by Lemma 2 where

$$
\begin{aligned}
\mathcal{M}_{241} &= -2T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-3}\sum\nolimits_{i_1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z)\,q_{jt}^{-3}\mathbb{E}\left[\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2\right|\mathcal{F}_t] \\
\mathcal{M}_{242} &= -2T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-3}\sum\nolimits_{i_1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z)\,q_{jt}^{-3}\left\{\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2-\mathbb{E}\left[\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2\right|\mathcal{F}_t]\right\} \\
\mathcal{M}_{243} &= -2T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-3}\sum\nolimits_{i_1\neq i_2}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z)\,q_{jt}^{-3}(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})\,\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2.
\end{aligned}
$$

Note that $\mathcal{M}_{241}=O_p\left(J^{2d_z}n^{-2}T^{-1}\right)$. For $\mathcal{M}_{242}$ it is mean zero with variance,

$$
\begin{aligned}
\mathbb{E}\left|\mathcal{M}_{242}\right|^2 &= 4T^{-4}\sum\nolimits_{t=1}^{T} n_t^{-6}\sum\nolimits_{i_1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbb{I}_{jt}(z)\,q_{jt}^{-6}\left\{\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2-\mathbb{E}\left[\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2\right|\mathcal{F}_t]\right\}^2\right] \\
&\leq CT^{-4}\sum\nolimits_{t=1}^{T} n_t^{-6}\sum\nolimits_{i_1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbb{I}_{jt}(z)\,q_{jt}^{-5}\right] \\
&\leq CT^{-3}n^{-5}J^{5d},
\end{aligned}
$$

and so $\mathcal{M}_{242}=o_p\left(J^{5d_z/2}n^{-5/2}T^{-3/2}\right)=o_p\left(J^{2d_z}n^{-2}T^{-1}\right)$ by Markov's inequality and Assumption 3. Next, $\mathcal{M}_{243}$

$$
\mathcal{M}_{243}=-2T^{-2}\sum\nolimits_{t=1}^{T} n_t^{-3}\sum\nolimits_{i_1\neq i_2}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z)\,q_{jt}^{-3}(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})\,\mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^2,
$$

is conditionally mean zero with variance

$$
\begin{aligned}
\mathbb{E}\left[\left|\mathcal{M}_{243}\right|^2\right] &= 4T^{-4}\sum\nolimits_{t=1}^{T} n_t^{-6}\sum\nolimits_{\substack{i_1\neq i_2,\\ i_3\neq i_4}}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \\
&\qquad \mathbb{E}\left[\mathbb{I}_{jt}(z)\,q_{jt}^{-6}(\mathbb{I}_{jt}(z_{i_2t})-q_{jt})(\mathbb{I}_{jt}(z_{i_4t})-q_{jt})\,\mathbb{I}_{jt}(z_{i_1t})\,\mathbb{I}_{jt}(z_{i_3t})\,\sigma_{i_1t}^2\sigma_{i_3t}^2\right] \\
&= \mathcal{N}_{243}^{(1)}+\mathcal{N}_{243}^{(2)}+\mathcal{N}_{243}^{(3)},
\end{aligned}
$$

where

$$\mathcal{N}_{243}^{(1)} = 4T^{-4}\sum_{t=1}^{T} n_t^{-6} \sum_{\substack{i_1\neq i_2, i_3\neq i_4\\ i_1\neq i_3, i_2=i_4}}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbb{I}_{jt}(z)\, q_{jt}^{-6} \left(\mathbb{I}_{jt}(z_{i_2t}) - q_{jt}\right)\left(\mathbb{I}_{jt}(z_{i_4t}) - q_{jt}\right)\mathbb{I}_{jt}(z_{i_1t})\,\mathbb{I}_{jt}(z_{i_3t})\,\sigma_{i_1t}^2\sigma_{i_3t}^2\right]$$

$$\mathcal{N}_{243}^{(2)} = 4T^{-4}\sum_{t=1}^{T} n_t^{-6} \sum_{\substack{i_1\neq i_2, i_3\neq i_4\\ i_2=i_4, i_1=i_3}}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbb{I}_{jt}(z)\, q_{jt}^{-6} \left(\mathbb{I}_{jt}(z_{i_2t}) - q_{jt}\right)\left(\mathbb{I}_{jt}(z_{i_4t}) - q_{jt}\right)\mathbb{I}_{jt}(z_{i_1t})\,\mathbb{I}_{jt}(z_{i_3t})\,\sigma_{i_1t}^2\sigma_{i_3t}^2\right]$$

$$\mathcal{N}_{243}^{(3)} = 4T^{-4}\sum_{t=1}^{T} n_t^{-6} \sum_{\substack{i_1\neq i_2, i_3\neq i_4\\ i_1=i_4, i_2=i_3}}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbb{I}_{jt}(z)\, q_{jt}^{-6} \left(\mathbb{I}_{jt}(z_{i_2t}) - q_{jt}\right)\left(\mathbb{I}_{jt}(z_{i_4t}) - q_{jt}\right)\mathbb{I}_{jt}(z_{i_1t})\,\mathbb{I}_{jt}(z_{i_3t})\,\sigma_{i_1t}^2\sigma_{i_3t}^2\right].$$

Then,

$$\begin{aligned}\mathcal{N}_{243}^{(1)} &= 4T^{-4}\sum_{t=1}^{T} n_t^{-6} \sum_{\substack{i_1\neq i_2, i_3\neq i_2\\ i_1\neq i_3}}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbb{I}_{jt}(z)\, q_{jt}^{-6} \left(\mathbb{I}_{jt}(z_{i_2t}) - q_{jt}\right)^2 \mathbb{I}_{jt}(z_{i_1t})\,\mathbb{I}_{jt}(z_{i_3t})\,\sigma_{i_1t}^2\sigma_{i_3t}^2\right]\\ &\leq CT^{-3}n^{-3}J^{3d_z}\end{aligned}$$

Next

$$\begin{aligned}\mathcal{N}_{243}^{(2)} &= 4T^{-4}\sum_{t=1}^{T} n_t^{-6} \sum_{i_1\neq i_2}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbb{I}_{jt}(z)\, q_{jt}^{-6} \left(\mathbb{I}_{jt}(z_{i_2t}) - q_{jt}\right)^2 \mathbb{I}_{jt}(z_{i_1t})\,\sigma_{i_1t}^4\right]\\ &\leq CT^{-3}n^{-4}J^{4d_z},\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_{243}^{(3)} &= 4T^{-4}\sum_{t=1}^{T} n_t^{-6} \sum_{i_1\neq i_2}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbb{I}_{jt}(z)\, q_{jt}^{-6} \left(\mathbb{I}_{jt}(z_{i_2t}) - q_{jt}\right)\left(\mathbb{I}_{jt}(z_{i_1t}) - q_{jt}\right)\mathbb{I}_{jt}(z_{i_1t})\,\mathbb{I}_{jt}(z_{i_2t})\,\sigma_{i_1t}^2\sigma_{i_2t}^2\right]\\ &\leq CT^{-3}n^{-4}J^{4d_z}.\end{aligned}$$

Thus, $\mathcal{M}_{243}$ is mean zero and of order $O_p\left(T^{-3/2}n^{-3/2}J^{3dz/2}\right)$.

The next cross product term is

$$\begin{aligned}\mathcal{M}_{25} &= \mathbb{E}\left[\mathcal{L}_{21}\mathcal{L}_{23}\middle|\,\mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right]\\ &= T^{-2}\sum_{t=1}^{T} n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}q_{jt}^{-3}\left(\tilde{q}_{jt}-q_{jt}\right)^2\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2\\ &= \mathbb{E}\left[\left|\mathcal{L}_{22}\right|^2\middle|\,\mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right]\\ &\quad -T^{-2}\sum_{t=1}^{T} n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,q_{jt}^{-4}\tilde{q}_{jt}^{-1}\left(\tilde{q}_{jt}-q_{jt}\right)^3\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2.\end{aligned}$$

The second term satisfies

$$\begin{aligned}&\left|T^{-2}\sum_{t=1}^{T} n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,q_{jt}^{-4}\tilde{q}_{jt}^{-1}\left(\tilde{q}_{jt}-q_{jt}\right)^3\mathbb{I}_{jt}(z_{it})\,\sigma_{it}^2\right|\\ \leq\;& CT^{-2}\sum_{t=1}^{T} n_t^{-1}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,q_{jt}^{-4}\left|\tilde{q}_{jt}-q_{jt}\right|^3\\ \leq\;& C\frac{J^{2d}}{n^2T}\left(\frac{J^{d/2}\log\left(J^d\vee T\right)^{3/2}}{n^{1/2}}\right),\end{aligned}$$

and so

$$\mathcal{M}_{25} = \mathbb{E}\left[|\mathcal{L}_{22}|^2 \right| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T \Big] + o_p\left(J^{2d_z} n^{-2} T^{-1} \right).$$

Finally,

$$\begin{aligned}
\mathcal{M}_{26} &= \mathbb{E}\left[\mathcal{L}_{22}\mathcal{L}_{23} \right| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T] \\
&= -T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) \tilde{q}_{jt}^{-1} q_{jt}^{-4} (\tilde{q}_{jt} - q_{jt})^3 \mathbb{I}_{jt}(z_{it}) \sigma_{it}^2.
\end{aligned}$$

But

$$\begin{aligned}
& \left| T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) \tilde{q}_{jt}^{-1} q_{jt}^{-4} (\tilde{q}_{jt} - q_{jt})^3 \mathbb{I}_{jt}(z_{it}) \sigma_{it}^2 \right| \\
\leq\ & C T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-1} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \mathbb{I}_{jt}(z) q_{jt}^{-4} (|\tilde{q}_{jt} - q_{jt}|)^3 \\
\leq\ & C T^{-1} \cdot \max_{1 \leq t \leq T} \max_{1 \leq j \leq J_t^{d_z}} q_{jt}^{-3} |\hat{q}_{jt} - q_{jt}|^3 \cdot J^{d_z} n^{-1},
\end{aligned}$$

so that $\mathcal{M}_{26} = O_p\left(T^{-1} n^{-5/2} J^{5d_z/2} \log(J \vee T)^{3/2} \right) = o_p\left(J^{2d_z} n^{-2} T^{-1} \right)$ by Lemma 2 and Assumption 3. Thus,

$$\begin{aligned}
\mathcal{M}_2 &= \mathbb{E}\left[(\mathcal{L}_{21} + \mathcal{L}_{22} + \mathcal{L}_{23})^2 \right| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T \Big] \\
&= 3\mathbb{E}\left[|\mathcal{L}_{22}|^2 \right| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T \Big] + 2\mathbb{E}\left[\mathcal{L}_{21}\mathcal{L}_{22} \right| \mathfrak{Z}, \mathfrak{X}, \mathcal{F}_1, \ldots, \mathcal{F}_T] + o_p\left(\frac{J^{2d_z}}{nT} \right) \\
&= T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z) q_{jt}^{-2} \mathbb{E}\left[\mathbb{I}_{jt}(z_{it}) \sigma_{it}^2 \right| \mathcal{F}_t] \\
&\quad + T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-3} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z) q_{jt}^{-3} \mathbb{E}\left[\mathbb{I}_{jt}(z_{it}) \sigma_{it}^2 \right| \mathcal{F}_t] \\
&\quad + T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z) q_{jt}^{-2} \left\{ \mathbb{I}_{jt}(z_{it}) \sigma_{it}^2 - \mathbb{E}\left[\mathbb{I}_{jt}(z_{it}) \sigma_{it}^2 \right| \mathcal{F}_t] \right\} \\
&\quad - 2T^{-2} \sum\nolimits_{t=1}^{T} n_t^{-3} \sum\nolimits_{i_1 \neq i_2}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z) q_{jt}^{-3} (\mathbb{I}_{jt}(z_{i_2 t}) - q_{jt}) \mathbb{I}_{jt}(z_{i_1 t}) \sigma_{i_1 t}^2 \\
&= \frac{J^{d_z}}{nT} \sum\nolimits_{t=1}^{T} \mathcal{V}_t^{(1)}(z) + \frac{J^{2d_z}}{n^2 T} \sum\nolimits_{t=1}^{T} \mathcal{V}_t^{(2)}(z) + \frac{J^{3d_z/2}}{n^{3/2} T^{3/2}} \sum\nolimits_{t=1}^{T} \mathcal{C}_t(z),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{V}_t^{(1)}(z) &= n J^{-d_z} T^{-1} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z) q_{jt}^{-2} \mathbb{E}\left[\mathbb{I}_{jt}(z_{it}) \sigma_{it}^2 \right| \mathcal{F}_t], \\
\mathcal{V}_t^{(2)}(z) &= n^2 J^{-2d_z} T^{-1} n_t^{-3} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z) q_{jt}^{-3} \mathbb{E}\left[\mathbb{I}_{jt}(z_{it}) \sigma_{it}^2 \right| \mathcal{F}_t] \\
\mathcal{C}_t(z) &= n^{3/2} J^{-3d_z/2} T^{-1/2} n_t^{-2} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z) q_{jt}^{-2} \left\{ \mathbb{I}_{jt}(z_{it}) \sigma_{it}^2 - \mathbb{E}\left[\mathbb{I}_{jt}(z_{it}) \sigma_{it}^2 \right| \mathcal{F}_t] \right\} \\
&\quad - 2n^{3/2} J^{-3d_z/2} T^{-1/2} n_t^{-3} \sum\nolimits_{i_1 \neq i_2}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{I}_{jt}(z) q_{jt}^{-3} (\mathbb{I}_{jt}(z_{i_2 t}) - q_{jt}) \mathbb{I}_{jt}(z_{i_1 t}) \sigma_{i_1 t}^2.
\end{aligned}$$

B.7.3 Term: $\mathcal{M}_3$

We have,

$$
\begin{aligned}
\mathcal{M}_3 &= \mathbb{E}\left[\left|T^{-1}\sum\nolimits_{t=1}^{T} n_t^{-1}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\,x_{it}'\left(\hat{\beta}_t-\beta_t\right)\right|^2 \middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] \\
&= \mathbb{E}\left[\left|T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}\hat{h}_t(z)'\left(\hat{\beta}_t-\beta_t\right)\right|^2 \middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] \\
&\leq 2\mathbb{E}\left[\left|T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}h_t(z)'\left(\hat{\beta}_t-\beta_t\right)\right|^2 \middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] \\
&\quad +2\mathbb{E}\left[\left|T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}\left(\hat{h}_t(z)-h_t(z)\right)'\left(\hat{\beta}_t-\beta_t\right)\right|^2 \middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] \\
&= \mathcal{M}_{31}+\mathcal{M}_{32}
\end{aligned}
$$

However, following similar steps as in the proof of Lemma 4

$$
\mathcal{M}_{31} = O_p\left(n^{-1}T^{-1}\right)+O_p\left(J^{-4}\right)+O_p\left(J^{2d_z}n^{-3}\right)+O_p\left(J^{d_z-4}n^{-2}\right).
$$

The $O_p\left(n^{-1}T^{-1}\right)$ term is not a function of J and the remaining terms are $o_p\left(J^{2d_z}n^{-2}T^{-1}\right)$. By the CS inequality, the second term satisfies

$$
\mathcal{M}_{32} \leq T^{-1}\sum\nolimits_{t=1}^{T}\left\|\hat{h}_t(z)-h_t(z)\right\|^2 \times T^{-1}\sum\nolimits_{t=1}^{T}\mathbb{E}\left[\left\|\hat{\beta}_t-\beta_t\right\|^2 \middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right].
$$

The first factor is $O_p\left(n^{-1}J^{d_z}\right)$ by similar steps as in the proof of Theorem 1 and the second factor, following similar steps as in the proof of Lemma 4, so that

$$
\mathcal{M}_{32} = O_p\left(n^{-1}J^{d_z}\right)\times O_p\left(J^{-4}+n^{-1}J^{-2}\right) = o_p\left(J^{2d_z}n^{-2}T^{-1}\right),
$$

and the result follows.

B.7.4 Term: $\mathcal{M}_4$

We have,

$$
\begin{aligned}
\mathcal{M}_4 &= 2\mathcal{L}_1\mathbb{E}\left[\mathcal{L}_3 \middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] \\
&= 2\mathcal{L}_1\mathbb{E}\left[T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}\hat{h}_t(z)'\,\hat{\Omega}_{\mathrm{uu},t}^{-1}\,X_t'M_{B_t}\left(\mu(z_t)+\varepsilon_t\right)/\,n_t \middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] \\
&= 2\mathcal{L}_1\times T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}\hat{h}_t(z)'\,\hat{\Omega}_{\mathrm{uu},t}^{-1}\,X_t'M_{B_t}\mu(z_t)/\,n_t,
\end{aligned}
$$

where, with some abuse of notation, we define $\mu(z_t)$ as the $n_t \times 1$ vector $\mu(z_t) = (\mu(z_1),\mu(z_2),\ldots,\mu(z_{n_t}))'$. The first factor is $O_p\left(J^{-1}\right)$ by the proof of Theorem 1 and the second factor satisfies

$$
T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}\hat{h}_t(z)'\,\hat{\Omega}_{\mathrm{uu},t}^{-1}\,\mathbf{X}_t'M_{B_t}\mu(z_t)/\,n_t = \mathcal{M}_{41}+\mathcal{M}_{42},
$$

where

$$
\begin{aligned}
\mathcal{M}_{41} &= T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}h_t(z)'\,\hat{\Omega}_{\mathrm{uu},t}^{-1}\,(H_t+U_t)'\,M_{B_t}\mu(z_t)/\,n_t \\
\mathcal{M}_{42} &= T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}\left(\hat{h}_t(z)-h_t(z)\right)'\,\hat{\Omega}_{\mathrm{uu},t}^{-1}\,X_t'M_{B_t}\mu(z_t)/\,n_t.
\end{aligned}
$$

Following similar steps as in the proof of Lemma 4 we have that,

$$|\mathcal{M}_{41}|^2 = O_p\left(J^{-4}\right) + O_p\left(n^{-1}T^{-1}J^{-2}\right) + O_p\left(n^{-2}J^{-2}\right) + O_p\left(n^{-1}J^{-6}\right) + O_p\left(J^{2d_z-2}n^{-3}\right).$$

For $\mathcal{M}_{42}$, by the CS inequality,

$$|\mathcal{M}_{42}|^2 \leq T^{-1}\sum_{t=1}^{T}\left\|\hat{h}_t(z) - h_t(z)\right\|^2 \times T^{-1}\sum_{t=1}^{T}\mathbf{1}_{\beta,t}\left\|\hat{\Omega}_{\mathrm{uu},t}^{-1}\, X_t' M_{B_t}\mu(z_t)/\, n_t\right\|^2.$$

The first factor is $O_p\left(n^{-1}J^{d_z}\right)$ by the same steps as in the proof of Theorem 1. The second factor is

$$\begin{aligned} T^{-1}\sum_{t=1}^{T}\mathbf{1}_{\beta,t}\left\|\hat{\Omega}_{\mathrm{uu},t}^{-1}\, X_t' M_{B_t}\mu(z_t)/\, n_t\right\|^2 &\leq T^{-1}\sum_{t=1}^{T}\mathbf{1}_{\beta,t}\lambda_{\max}\left(\hat{\Omega}_{\mathrm{uu},t}^{-1}\right)^2\left\| X_t' M_{B_t}\mu(z_t)/\, n_t\right\|^2 \\ &\leq CT^{-1}\sum_{t=1}^{T}\left\| X_t' M_{B_t}\mu(z_t)/\, n_t\right\|^2. \end{aligned}$$

Following similar steps as in the proof of Lemma 4 we have that,

$$|\mathcal{M}_{42}|^2 = O_p\left(n^{-1}J^{d_z}\right) O_p\left(J^{-4} + n^{-1}J^{-2}\right) = o_p\left(J^{2d_z}n^{-2}T^{-1}\right).$$

Thus,

$$\begin{aligned} |\mathcal{M}_4|^2 &= O_p\left(J^{-6}\right) + O_p\left(n^{-1}T^{-1}J^{-4}\right) + O_p\left(n^{-2}J^{-4}\right) + O_p\left(n^{-1}J^{-8}\right) + O_p\left(J^{2d_z-4}n^{-3}\right) \\ &\quad + O_p\left(n^{-1}J^{d_z}J^{-6}\right) + O_p\left(J^{d_z-4}n^{-2}\right), \end{aligned}$$

so that $\mathcal{M}_4 = o_p\left(J^{-2} + J^{2d_z}n^{-2}T^{-1}\right)$ by Assumption 3.

B.7.5 Term: $\mathcal{M}_5$

Finally, we have

$$\begin{aligned} \mathcal{M}_5 &= 2\mathbb{E}\left[(\mathcal{L}_{21}+\mathcal{L}_{22}+\mathcal{L}_{23})\,\mathcal{L}_3 \middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] \\ &= 2\mathbb{E}\left[T^{-2}\sum_{t=1}^{T} n_t^{-2}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\,\varepsilon_{it}x_{it}'\left(\hat{\beta}_t - \beta_t\right)\middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] \\ &= 2\mathbb{E}\left[T^{-2}\sum_{t=1}^{T} n_t^{-2}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\,\varepsilon_{it}x_{it}'\hat{\Omega}_{\mathrm{uu},t}^{-1}\, X_t' M_{B_t}(\mu(z_t)+\varepsilon_t)/\, n_t\middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] \\ &= 2T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\,\iota_i'\mathbb{E}\left[\varepsilon_t\varepsilon_t'\middle| \mathfrak{Z},\mathfrak{X},\mathcal{F}_1,\ldots,\mathcal{F}_T\right] M_{B_t}X_t\hat{\Omega}_{\mathrm{uu},t}^{-1}x_{it} \\ &= 2T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\,\iota_i'\Sigma_t M_{B_t}(H_t+U_t)\,\hat{\Omega}_{\mathrm{uu},t}^{-1}x_{it} \\ &= \mathcal{M}_{51} + \mathcal{M}_{52}, \end{aligned}$$

where $\Sigma_t = \mathrm{diag}\left(\sigma_{1t}^2,\ldots,\sigma_{n_t t}^2\right)$ and

$$\begin{aligned} \mathcal{M}_{51} &= 2T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\,\iota_i'\Sigma_t M_{B_t}H_t\hat{\Omega}_{\mathrm{uu},t}^{-1}x_{it} \\ \mathcal{M}_{52} &= 2T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\,\iota_i'\Sigma_t M_{B_t}U_t\hat{\Omega}_{\mathrm{uu},t}^{-1}x_{it} \end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{51} &= 2T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\left|\iota_i'\Sigma_t M_{B_t}\left(H_t - B_t\Pi_t^0\right)\hat{\Omega}_{\mathrm{uu},t}^{-1}x_{it}\right| \\
&\le 2T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\left\|\iota_i'\Sigma_t\right\|\left\|M_{B_t}\left(H_t - B_t\Pi_t^0\right)\right\|\left\|\hat{\Omega}_{\mathrm{uu},t}^{-1}x_{it}\right\| \\
&\le \max_{1\le t\le T}\left\|H_t - B_t\Pi_t^0\right\| CJ^d T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\mathbb{I}_{jt}(z_{it})\left\|x_{it}\right\|
\end{aligned}$$

However,

$$\begin{aligned}
\max_{1\le t\le T}\left\|H_t - B_t\Pi_t^0\right\|^2 &= \max_{1\le t\le T}\mathrm{tr}\left(\left(H_t - B_t\Pi_t^0\right)'\left(H_t - B_t\Pi_t^0\right)\right) \\
&= \max_{1\le t\le T}\sum_{i=1}^{n}\left\|h_t(z_{it}) - B_t(z_{it})'\pi_t^0\right\|^2 \\
&= \sum_{i=1}^{n_t}\sum_{\ell=1}^{d_x}\max_{1\le t\le T}\left|\sum_{j=1}^{J_t^{d_z}}\widehat{\mathbb{I}}_{jt}(z_{it})\,h_{t,\ell}(z_{it}) - \widehat{\mathbb{I}}_{jt}(z_{it})\,\pi_{jt,\ell}^0\right|^2 \\
&\le \sum_{i=1}^{n_t}\sum_{\ell=1}^{d_x}\max_{1\le t\le T}\max_{1\le j\le J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,h_{t,\ell}(z) - \widehat{\mathbb{I}}_{jt}(z)\,\pi_{jt,\ell}^0\right|^2,
\end{aligned}$$

so that $\max_{1\le t\le T}\left\|H_t - B_t\Pi_t^0\right\|^2 = O_p\left(nJ^{-2}\right)$ so that, by Markov's inequality, $\mathcal{M}_{51} = O_p\left(J^{-1}T^{-1}n^{-3/2}\right) = o_p\left(J^{2d_z}n^{-2}T^{-1}\right)$ by Assumption 3.

$$\mathcal{M}_{52} = 2T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\,\iota_i'\Sigma_t M_{B_t}U_t\hat{\Omega}_{\mathrm{uu},t}^{-1}x_{it}$$

But

$$\begin{aligned}
\left|\mathcal{M}_{52}\right| &\le 2T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\tilde{q}_{jt}^{-1}\mathbb{I}_{jt}(z_{it})\left|\iota_i'\Sigma_t M_{B_t}U_t\hat{\Omega}_{\mathrm{uu},t}^{-1}x_{it}\right| \\
&\le CJ^{d_z}T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\mathbb{I}_{jt}(z_{it})\left\|U_t\right\|\left\|x_{it}\right\|,
\end{aligned}$$

and

$$\begin{aligned}
& J^{d_z}T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbb{E}\left[\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\mathbb{I}_{jt}(z_{it})\,\mathbb{E}\left[\left\|U_t\right\|\left\|x_{it}\right\|\middle| z_t,\mathcal{F}_t\right]\right] \\
\le\; & J^{d_z}T^{-2}\sum_{t=1}^{T} n_t^{-3}\sum_{i_1=1}^{n_t}\sum_{j=1}^{J_t^{d_z}}\mathbb{E}\left[\mathbf{1}_{jt}\mathbb{I}_{jt}(z)\,\mathbb{I}_{jt}(z_{it})\sqrt{\mathbb{E}\left[\left\|U_t\right\|^2\middle| z_t,\mathcal{F}_t\right]\mathbb{E}\left[\left\|x_{it}\right\|^2\middle| z_t,\mathcal{F}_t\right]}\right] \\
\le\; & CT^{-1}n^{-3/2}.
\end{aligned}$$

Thus, $\mathcal{M}_{52} = O_p\left(T^{-1}n^{-3/2}\right)$ which is $o_p\left(J^{2d_z}n^{-2}T^{-1}\right)$ under Assumption 3.

B.8 Proofs of Lemmas

Proof of Lemma 1. We would like to show that there exists a γ_{jt}^0 such that

$$\max_{1\le t\le T}\max_{1\le j\le J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^0\right|=O_p\left(J^{-1}\right).$$

Let $\gamma_{jt}^0=\gamma_{jt}^0(z_{1t},\ldots,z_{n_tt})=\mu\left(\hat{b}_t\right)$ where $\hat{b}_t=\left(\hat{b}_{(j_1-1/2)t,1},\ldots,\hat{b}_{(j_{d_z}-1/2)t,d_z}\right)'$. We have

$$\mathbb{P}\left(\max_{1\le t\le T}\max_{1\le j\le J}\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^0\right|>\frac{C}{J}\right)\le\sum\nolimits_{t=1}^{T}\sum\nolimits_{j=1}^{J}\mathbb{P}\left(\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^0\right|>\frac{C}{J}\right).$$

Let us focus on the summand,

$$\begin{aligned}&\mathbb{P}\left(\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^0\right|>\frac{C}{J}\right|z_{1t},\ldots,z_{n_tt}\Bigg)\\ =\;&\mathbb{P}\left(\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\sum\nolimits_{s=1}^{d_z}\frac{\partial\mu(z)}{\partial z_s}\bigg|_{z=\tilde{z}_{jt}}\left(z_s-\hat{b}_{(j_s-1/2)t,s}\right)\right|>\frac{C}{J}\right|z_{1t},\ldots,z_{n_tt}\Bigg)\end{aligned}$$

where $\tilde{z}_{jt}=\alpha z+(1-\alpha)\,\hat{b}_t$, $\alpha\in(0,1)$. Now, choose C_1 sufficiently large such that $\max_{1\le s\le d_z}\sup_z\left|\frac{\partial\mu(z)}{\partial z_s}\Big|_{z=z}\right|<C_1$. Then,

$$\begin{aligned}&\mathbb{P}\left(\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\sum\nolimits_{s=1}^{d_z}\frac{\partial\mu(z)}{\partial z_s}\bigg|_{z=\tilde{z}_{jt}}\left(z_s-\hat{b}_{(j_s-1/2)t,s}\right)\right|>\frac{C}{J}\right|z_{1t},\ldots,z_{n_tt}\Bigg)\\ \le\;&\mathbb{P}\left(\sup_z\sum\nolimits_{s=1}^{d_z}\left|z_s-\hat{b}_{(j_s-1/2)t,s}\right|\widehat{\mathbb{I}}_{jt}(z)>\frac{C_1}{J}\right|z_{1t},\ldots,z_{n_tt}\Bigg)\\ \le\;&\sum\nolimits_{s=1}^{d_z}\mathbb{P}\left(\left(\hat{b}_{j_st,s}-\hat{b}_{(j_s-1)t,s}\right)>\frac{C_1}{J}\right|z_{1t},\ldots,z_{n_tt}\Bigg)\end{aligned}$$

Thus, we can focus on

$$\mathbb{P}\left(\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^0\right|>\frac{C}{J}\right|\Bigg)\le\sum\nolimits_{s=1}^{d_z}\mathbb{P}\left(\left(\hat{b}_{j_st,s}-\hat{b}_{(j_s-1)t,s}\right)>\frac{C_1}{J}\right).$$

Recall that we can define the empirical quantile function in terms of the order statistics of the z_{it}'s:

$$\hat{F}_{t,s}^{-1}(p)=z_{(k)t,s},\qquad\frac{k-1}{n}<p\le\frac{k}{n},$$

where $z_{(k)t,s}$ is the kth order statistic of $z_{it,s}$. Thus we have

$$\begin{aligned}&\mathbb{P}\left(\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^0\right|>\frac{C}{J}\right|\Bigg)\\ \le\;&\sum\nolimits_{s=1}^{d_z}\mathbb{P}\left(\left(\hat{b}_{j_st,s}-\hat{b}_{(j_s-1)t,s}\right)>\frac{C_1}{J}\right)\\ =\;&\sum\nolimits_{s=1}^{d_z}\mathbb{P}\left(\left(z_{(k_{2,s})t,s}-z_{(k_{1,s})t,s}\right)>\frac{C_1}{J}\right)\\ =\;&\sum\nolimits_{s=1}^{d_z}\mathbb{E}\left[\mathbb{P}\left(F_{z_{it,s}|\mathcal{F}_t}^{-1}\left(u_{(k_{2,s})t,s}\right)-F_{z_{it,s}|\mathcal{F}_t}^{-1}\left(u_{(k_{1,s})t,s}\right)>\frac{C_1}{J}\right|\mathcal{F}_t\Bigg)\right]\end{aligned}$$

where

$$\frac{k_{1,s}-1}{n}<\frac{j_s-1}{J}\le\frac{k_{1,s}}{n},\qquad\frac{k_{2,s}-1}{n}<\frac{j_s}{J}\le\frac{k_{2,s}}{n},$$

for $s=1,\ldots,d_z$ and $u_{(k)t,s}$ are order statistics of (conditionally) independent standard uniform random variables. The last line uses the fact that, conditional of $\mathcal{F}_t$, z_{it} are iid with CDF $F_{z_{it}|\mathcal{F}_t}(z)$ and so we can use the probability integral transform to map to the uniform order statistics from a sample of n_t standard uniform random variables (conditional on $\mathcal{F}_t$). Using another mean-value expansion we have

$$\begin{aligned}
& \mathbb{P}\left(F^{-1}_{z_{it,s}|\mathcal{F}_t}\left(u_{(k_{2,s})t,s}\right)-F^{-1}_{z_{it,s}|\mathcal{F}_t}\left(u_{(k_{1,s})t,s}\right)>\frac{C_1}{J}\middle|\mathcal{F}_t\right)\\
= & \mathbb{P}\left(q_{z_{it,s}|\mathcal{F}_t}\left(\tilde{u}\right)\left(u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\right)>\frac{C_1}{J}\middle|\mathcal{F}_t\right)\\
= & \mathbb{P}\left(q_{z_{it,s}|\mathcal{F}_t}\left(\tilde{u}\right)\left(u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}-\mathbb{E}\left[u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]\right)>\frac{C_1}{J}-q_{z_{it,s}|\mathcal{F}_t}\left(\tilde{u}\right)\mathbb{E}\left[u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_t\right),
\end{aligned}$$

where $\tilde{u}=\alpha u_{(k_{1,s})t,s}+(1-\alpha)\,u_{(k_{2,s})t,s}$ and $\alpha\in(0,1)$, and $q_{z_{it,s}|\mathcal{F}_t}(u)=\frac{\partial}{\partial u}F^{-1}_{z_{it,s}|\mathcal{F}_t}(u)=\frac{1}{f_{z_{it,s}|\mathcal{F}_t}\left(F^{-1}_{z_{it,s}|\mathcal{F}_t}(u)\right)}$.

Under our assumptions $q_{z_{it,s}|\mathcal{F}_t}(u)$ is positively bounded from above and below. Conditional on $\mathcal{F}_t$, $u_{(k)t,s}|\mathcal{F}_t\sim Beta\,(k,n+k-1)$ so that

$$\mathbb{E}\left[u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]=\frac{k_{2,s}-k_{1,s}}{n+1}.$$

The inequalities defining $k_{1,s}$ and $k_{2,s}$ imply that

$$\frac{1}{J}-\frac{1}{n}\leq\frac{k_{2,s}-k_{1,s}}{n}\leq\frac{1}{J}+\frac{1}{n}.$$

Thus,

$$\begin{aligned}
& \mathbb{P}\left(F^{-1}_{z_{it,s}|\mathcal{F}_t}\left(u_{(k_{2,s})t,s}\right)-F^{-1}_{z_{it,s}|\mathcal{F}_t}\left(u_{(k_{1,s})t,s}\right)>\frac{C_1}{J}\middle|\mathcal{F}_t\right)\\
= & \mathbb{P}\left(\left(u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}-\mathbb{E}\left[u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]\right)>\frac{1}{q_{z_{it,s}|\mathcal{F}_t}\left(\tilde{u}\right)}\frac{C_1}{J}-\mathbb{E}\left[u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_t\right)\\
\leq & \mathbb{P}\left(\left|u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}-\mathbb{E}\left[u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]\right|\right.\\
& \left.>\left(\frac{C_1}{J\cdot\mathbb{E}\left[u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]}-1\right)\mathbb{E}\left[u_{(k_{2,s})t,s}f-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_t\right)\\
\leq & \mathbb{P}\left(\left|u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}-\mathbb{E}\left[u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]\right|>\left(\frac{1}{\left(\frac{1}{J}-\frac{1}{n}\right)}\frac{C_1}{J}-1\right)\mathbb{E}\left[u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_t\right)\\
\leq & 2\exp\left\{-C_2\left(n+1\right)\left(\frac{1}{\left(\frac{1}{J}-\frac{1}{n}\right)}\frac{C_3}{J}-1\right)^2\right\},
\end{aligned}$$

where the last line uses the fact that, conditional of $\mathcal{F}_t$, $u_{(k_{2,s})t,s}-u_{(k_{1,s})t,s}$ are the sum of individual uniform spacings and are distributed as $\left(u_{(k_2)t}-u_{(k_1)t}\right)|\mathcal{F}_t\sim Beta\left(k_{2,s}-k_{1,s},n-1-\left(k_{2,s}-k_{1,s}\right)\right)$ and Bobkov and Ledoux (2016, Proposition B.10). We can put all this together to obtain

$$\begin{aligned}
\mathbb{P}\left(\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma^0_{jt}\right|>\frac{C}{J}\right) & \leq \sum_{t=1}^{T}\sum_{j=1}^{J}\mathbb{E}\left[1\wedge 2\exp\left\{-C_2\left(n+1\right)\left(\frac{1}{\left(\frac{1}{J}-\frac{1}{n}\right)}\frac{C_3}{J}-1\right)^2\right\}\right]\\
& \leq \mathbb{E}\left[1\wedge 2JT\exp\left\{-C_2\left(n+1\right)\left(\frac{1}{\left(\frac{1}{J}-\frac{1}{n}\right)}\frac{C_3}{J}-1\right)^2\right\}\right]\\
& = o(1)
\end{aligned}$$

under our assumptions. Next we would like to show,

$$\mathbb{E}\left[\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^0\right|^2\right]=O\left(J^{-2}\right).$$

This follows immediately from the previous result since,

$$\mathbb{E}\left[\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^0\right|^2\right]\leq C_1\cdot\mathbb{P}\left[\left(\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}(z)\,\mu(z)-\widehat{\mathbb{I}}_{jt}(z)\,\gamma_{jt}^0\right|^2>\frac{C_2}{J^2}\right)\right]+\frac{C_2}{J^2}.$$

□

Proof of Lemma 2. By the proof of Lemma 1 we have that

$$\mathbb{P}\left(\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\max_{1\leq s\leq d_x}\left|\hat{b}_{j_st,s}-\hat{b}_{(j_s-1)t,s}\right|>\frac{C}{J}\right|\right)=o(1).$$

By Einmahl and Ruymgaart (1987, Theorem 3.1) for all sequences $\delta_n=O(J^{-d_z})$,

$$\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\left|\hat{q}_{jt}-q_{jt}\right|^2=O_p\left(\frac{\log\left(J^{d_z}\vee T\right)}{J^{d_z}n}\right)$$

provided that

$$\frac{J^{d_z}\log(n)}{n}\to 0,\qquad\text{and,}\qquad\frac{J^{d_z}}{\log(n)}\to\infty,$$

which are satisfied under our assumptions. □

Proof of Lemma 3. We will first find the order of

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^T\left\|\hat{\Omega}_{\text{uu},t}-\Omega_{\text{uu},t}\right\|^2 &= \frac{1}{T}\sum_{t=1}^T\left\|\left(X_t'M_{B_t}X_t/n_t\right)-\Omega_{\text{uu},t}\right\|^2\\ &\leq C\cdot\frac{1}{T}\sum_{t=1}^T\left\|X_t'M_{B_t}X_t/n_t-U_tU_t'/n_t\right\|^2+C\cdot\frac{1}{T}\sum_{t=1}^T\left\|U_tU_t'/n_t-\Omega_{\text{uu},t}\right\|^2.\end{aligned}$$

For the second term we have that

$$\begin{aligned}\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^T\left\|U_tU_t'/n_t-\Omega_{\text{uu},t}\right\|^2\right] &= \frac{1}{T}\sum_{t=1}^T\mathbb{E}\left\|\frac{1}{n_t}\sum_{i=1}^{n_t}u_{it}u_{it}'-\Omega_{\text{uu},t}\right\|^2\\ &= \frac{1}{T}\sum_{t=1}^T\operatorname{tr}\left(\frac{1}{n_t^2}\sum_{i_1,i_2}\mathbb{E}\left[\left(u_{i_1t}u_{i_1t}'-\Omega_{\text{uu},t}\right)'\left(u_{i_2t}u_{i_2t}'-\Omega_{\text{uu},t}\right)\right]\right)\\ &= \frac{1}{T}\sum_{t=1}^T\operatorname{tr}\left(\frac{1}{n_t^2}\sum_{i_1}\mathbb{E}\left[\left(u_{i_1t}u_{i_1t}'-\Omega_{\text{uu},t}\right)'\left(u_{i_1t}u_{i_1t}'-\Omega_{\text{uu},t}\right)\right]\right)\\ &\leq C\cdot\frac{1}{n},\end{aligned}$$

so this term is $O_p\left(n^{-1}\right)$ by Markov's inequality. For the first term note that,

$$\begin{aligned}X_t'M_{B_t}X_t/n_t-U_t'U_t/n_t &= (U_t+H_t)'M_{B_t}(U_t+H_t)/n_t\\ &= H_t'M_{B_t}H_t/n_t-U_t'(I_{n_t}-M_{B_t})U_t/n_t+U_t'M_{B_t}H_t/n_t+H_t'M_{B_t}U_t/n_t,\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{T}\sum\nolimits_{t=1}^{T}\left\| X_t' M_{B_t} X_t / n_t - U_t U_t' / n_t \right\|^2 &\leq C\cdot\frac{1}{T}\sum\nolimits_{t=1}^{T}\left\| H_t' M_{B_t} H_t / n_t \right\|^2 \\ &\quad + C\cdot\frac{1}{T}\sum\nolimits_{t=1}^{T}\left\| U_t' \left(I_{n_t} - M_{B_t}\right) U_t / n_t \right\|^2 \\ &\quad + C\cdot\frac{1}{T}\sum\nolimits_{t=1}^{T}\left\| U_t' M_{B_t} H_t / n_t \right\|^2\end{aligned}$$

For the first term,

$$\begin{aligned}\left\| H_t' M_{B_t} H_t / n_t \right\| &= n_t^{-1}\left\| \left(H_t - B_t \Pi_t^0\right)' M_{B_t}\left(H_t - B_t \Pi_t^0\right)\right\| \\ &\leq C\cdot n^{-1}\sum\nolimits_{i=1}^{n}\left\| h_t\left(z_{it}\right) - B_t\left(z_{it}\right)' \pi_t^0 \right\|^2 \\ &= C\cdot n^{-1}\sum\nolimits_{i=1}^{n}\sum\nolimits_{\ell=1}^{d_x}\left|\sum\nolimits_{j=1}^{J_t^{d_z}}\widehat{\mathbb{I}}_{jt}\left(z_{it}\right) h_{t,\ell}\left(z_{it}\right) - \widehat{\mathbb{I}}_{jt}\left(z_{it}\right)\pi_{jt,\ell}^0\right|^2 \\ &\leq C\cdot n^{-1}\sum\nolimits_{i=1}^{n}\sum\nolimits_{\ell=1}^{d_x}\max_{1\leq j\leq J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}(z) h_{t,\ell}(z) - \widehat{\mathbb{I}}_{jt}(z)\pi_{jt,\ell}^0\right|^2\sum\nolimits_{j=1}^{J_t^{d_z}}\widehat{\mathbb{I}}_{jt}\left(z_{it}\right) \\ &\leq C\cdot\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}(z) h_{t,\ell}(z) - \widehat{\mathbb{I}}_{jt}(z)\pi_{jt,\ell}^0\right|^2,\end{aligned}$$

and so $\left\| H_t' M_{B_t} H_t / n_t \right\| = O_p\left(J^{-2}\right)$ by Lemma 1. Next let $P_{B_t} = I_{n_t} - M_{B_t}$ with elements $\left[P_{B_t}\right]_{i,j} = p_{it,j}$ and note that,

$$\begin{aligned}\left\| U_t' P_{B_t} U_t / n_t \right\|^2 &= n_t^{-2}\mathbb{E}\left[\operatorname{tr}\left(\left(U_t' P_{B_t} U_t\right)' U_t' P_{B_t} U_t\right)\right] \\ &= n_t^{-2}\mathbb{E}\left[\operatorname{tr}\left(P_{B_t} U_t U_t' P_{B_t} U_t U_t'\right)\right] \\ &= n_t^{-2}\mathbb{E}\left[\operatorname{tr}\left(P_{B_t}\mathbb{E}\left[U_t U_t' P_{B_t} U_t U_t' \middle| z_t, \mathcal{F}_t\right]\right)\right].\end{aligned}$$

Then,

$$\begin{aligned}&\operatorname{tr}\left(P_{B_t}\mathbb{E}\left[U_t U_t' P_{B_t} U_t U_t' \middle| z_t, \mathcal{F}_t\right]\right) \\ =\ &\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{\ell_0=1}^{n_t}\sum\nolimits_{\ell_1=1}^{d_x}\sum\nolimits_{\ell_2=1}^{n_t}\sum\nolimits_{\ell_3=1}^{n_t}\sum\nolimits_{\ell_4=1}^{d_x} p_{it,\ell_0} p_{\ell_2 t,\ell_3}\mathbb{E}\left[u_{\ell_0 t,\ell_1} u_{\ell_2 t,\ell_1} u_{\ell_3 t,\ell_4} u_{it,\ell_4} \middle| z_t\right].\end{aligned}$$

This expectation is nonzero only when $\{\ell_0 = \ell_2, \ell_3 = i\}$ or $\{\ell_0 = \ell_3, \ell_2 = i\}$ or $\{\ell_0 = i, \ell_2 = \ell_3\}$. These correspond to the following three terms:

$$\begin{aligned}&\operatorname{tr}\left(P_{B_t}\mathbb{E}\left[U_t U_t' P_{B_t} U_t U_t' \middle| z_t, \mathcal{F}_t\right]\right) \\ \leq\ &\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{\ell_0=1}^{n_t}\sum\nolimits_{\ell_1=1}^{d_x}\sum\nolimits_{\ell_4=1}^{d_x} p_{it,\ell_0} p_{\ell_0 t,i}\mathbb{E}\left[u_{\ell_0 t,\ell_1}^2 u_{it,\ell_4}^2 \middle| z_t, \mathcal{F}_t\right] \\ &+\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{\ell_0=1}^{n_t}\sum\nolimits_{\ell_1=1}^{d_x}\sum\nolimits_{\ell_4=1}^{d_x} p_{it,\ell_0} p_{it,\ell_0}\mathbb{E}\left[u_{\ell_0 t,\ell_1} u_{\ell_0 t,\ell_4} u_{it,\ell_1} u_{it,\ell_4} \middle| z_t, \mathcal{F}_t\right] \\ &+\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{\ell_1=1}^{d_x}\sum\nolimits_{\ell_2=1}^{n_t}\sum\nolimits_{\ell_4=1}^{d_x} p_{it,i} p_{\ell_2 t,\ell_2}\mathbb{E}\left[u_{it,\ell_1} u_{it,\ell_4} u_{\ell_2 t,\ell_1} u_{\ell_2 t,\ell_4} \middle| z_t, \mathcal{F}_t\right] \\ \leq\ &C\cdot\operatorname{tr}\left(P_{B_t}^2\right) + C\cdot\operatorname{tr}\left(P_{B_t}^2\right) + C\cdot\left[\operatorname{tr}\left(P_{B_t}\right)\right]^2 \\ \leq\ &C\cdot\left(J^{d_z} + J^{2d_z}\right).\end{aligned}$$

Thus,

$$\frac{1}{T}\sum\nolimits_{t=1}^{T}\mathbb{E}\left\| U_t' P_{B_t} U_t / n_t \right\|^2 \leq C\cdot n^{-2}\left(J^{d_z} + J^{2d_z}\right),$$

so this term is $O_p\left(n^{-2} J^{2d_z}\right)$ by Markov's inequality. Finally, note that

$$\max_{1\leq t\leq T}\mathbb{E}\left[\left\| U_t' M_{B_t} H_t / n_t \right\|^2\right]$$

$$
\begin{aligned}
&\leq C \cdot n^{-2} \max_{1\leq t\leq T} \mathbb{E}\left[\left\| U_t' M_{B_t} H_t \right\|^2\right] \\
&= C \cdot n^{-2} \max_{1\leq t\leq T} \mathbb{E}\left[\operatorname{tr}\left(\mathbb{E}\left[U_t U_t' \middle| z_t, \mathcal{F}_t\right] M_{B_t} H_t H_t' M_{B_t}\right)\right] \\
&\leq C \cdot n^{-2} \max_{1\leq t\leq T} \mathbb{E}\left[\operatorname{tr}\left(\left(H_t - B_t \Pi_t^0\right)' M_{B_t}\left(H_t - B_t \Pi_t^0\right)\right)\right] \\
&\leq C \cdot n^{-2} \sum_{i=1}^{n} \sum_{\ell=1}^{d_x} \max_{1\leq t\leq T} \mathbb{E}\left[\left| h_{t,\ell}(z_{it}) - B_t(z_{it})' \pi_{t,\ell}^0 \right|^2\right] \\
&\leq C \cdot n^{-2} \sum_{i=1}^{n} \sum_{\ell=1}^{d_x} \max_{1\leq t\leq T} \mathbb{E}\left[\max_{1\leq j\leq J_t^{d_z}} \sup_z \left| \widehat{\mathbb{I}}_{jt}(z) h_{t,\ell}(z) - \widehat{\mathbb{I}}_{jt}(z) \pi_{jt,\ell}^0 \right|^2 \sum_{j=1}^{J_t^{d_z}} \widehat{\mathbb{I}}_{jt}(z)\right] \\
&\leq C \cdot n^{-1} J^{-2}.
\end{aligned}
$$

Thus,

$$
\frac{1}{T}\sum_{t=1}^{T} \left\| X_t' M_{B_t} X_t / n_t - U_t U_t' / n_t \right\|^2 = O_p\left(J^{-4}\right) + O_p\left(n^{-2} J^{2d_z}\right) + O_p\left(n^{-1} J^{-2}\right),
$$

and

$$
\frac{1}{T}\sum_{t=1}^{T} \left\| \hat{\Omega}_{\mathrm{uu},t} - \Omega_{\mathrm{uu},t} \right\|^2 = O_p\left(n^{-1}\right) + O_p\left(J^{-4}\right) + O_p\left(n^{-2} J^{2d_z}\right) + o_p\left(n^{-1}\right).
$$

We will now find the order of $\max_{1\leq t\leq T} \left\| \hat{\Omega}_{\mathrm{uu},t} - \Omega_{\mathrm{uu},t} \right\|$. Recall that

$$
\begin{aligned}
\hat{\Omega}_{\mathrm{uu},t} - \Omega_{\mathrm{uu},t} = \;& H_t' M_{B_t} H_t / n_t && \text{(B.2)} \\
& - U_t' \left(I_{n_t} - M_{B_t}\right) U_t / n_t && \text{(B.2)} \\
& + U_t' M_{B_t} H_t / n_t + H_t' M_{B_t} U_t / n_t && \text{(B.2)} \\
& + \left(U_t' U_t / n_t - \Omega_{\mathrm{uu},t}\right). && \text{(B.2)}
\end{aligned}
$$

The first term (equation (B.2)) satisfies $\max_{1\leq t\leq T} \left\| H_t' M_{B_t} H_t / n_t \right\| = O_p\left(J^{-2}\right)$ by the derivations above. Now consider equation (B.2). Let $a \in R^{d_x}$, $a \neq 0$. Then by the CS inequality it is sufficient to show that $\max_{1\leq t\leq T} \left| a' U_t' P_{B_t} U_t a \right| / n_t = O_p\left(J^{d_z} n^{-1}\right)$. First note that

$$
\max_{1\leq t\leq T} \left| a' U_t' P_{B_t} U_t a \right| \leq \max_{1\leq t\leq T} \left| a' U_t' P_{B_t} U_t a - a' \mathbb{E}\left[U_t' P_{B_t} U_t \middle| z_t, \mathcal{F}_t \right] a \right| + \max_{1\leq t\leq T} \left| a' \mathbb{E}\left[U_t' P_{B_t} U_t \middle| z_t, \mathcal{F}_t \right] a \right|.
$$

Define $\tilde{U}_t = U_t a$ which is a $n_t \times 1$ vector and note that conditional on $\mathcal{F}_t$, the elements of $\tilde{U}_t$ are independent (and mean zero). We will deal with the second term first,

$$
\begin{aligned}
\left| a' \mathbb{E}\left[U_t' P_{B_t} U_t \middle| z_t, \mathcal{F}_t \right] a \right| / n_t &= \left| \mathbb{E}\left[\operatorname{tr}\left(\tilde{U}_t' P_{B_t} \tilde{U}_t \right) \middle| z_t, \mathcal{F}_t \right] \right| / n_t \\
&\leq C n^{-1} \left| \operatorname{tr}\left(P_{B_t}\right) \right| \\
&\leq C n^{-1} J^{d_z}.
\end{aligned}
$$

By our assumption of sub-Gaussianity on x_{it} and that sums of sub-Gaussian variables are also sub-Gaussian, the Hanson-Wright inequality (see, for example, Rudelson and Vershynin (2013) yields

$$
\begin{aligned}
&\mathbb{P}\left(\left| \tilde{U}_t' P_{B_t} \tilde{U}_t - \mathbb{E}\left[\tilde{U}_t' P_{B_t} \tilde{U}_t \middle| z_t, \mathcal{F}_t \right] \right| \geq \delta \middle| z_t, \mathcal{F}_t \right) \\
\leq\;& 2\exp\left\{ -C \min\left(\frac{\delta^2}{2K^4 \left\| P_{B_t} \right\|^2}, \frac{\delta}{K^2 \sup_{\|y\|=1} \left\| P_{B_t} y \right\|} \right) \right\} \\
=\;& 2\exp\left\{ -C \min\left(\frac{\delta^2}{2K^4 J^{d_z}}, \frac{\delta}{K^2} \right) \right\},
\end{aligned}
$$

if we map $\delta \mapsto \delta \log(T)^{1/2} J^{d_z/2}$ we have that

$$\begin{aligned} & \mathbb{P}\left(J^{-d_z/2} \log(T)^{-1/2} \left| \tilde{U}_t' P_{B_t} \tilde{U}_t - \mathbb{E}\left[\tilde{U}_t' P_{B_t} \tilde{U}_t \right| z_t, \mathcal{F}_t \right] \right| \geq \delta \left| z_t, \mathcal{F}_t \right) \\ \leq \; & 2 \exp\left\{ -C \min\left(\frac{\delta^2 \log(T)}{2K^4}, \frac{\delta \log(T)^{1/2} J^{d_z/2}}{K^2} \right) \right\} \\ \leq \; & 2 \exp\left\{ -C \frac{\delta^2 \log(T)}{2K^4} \right\}, \end{aligned}$$

for sufficiently large n and T. Thus,

$$\begin{aligned} & \mathbb{P}\left(\max_{1 \leq t \leq T} J^{-d_z/2} \log(T)^{-1/2} \left| \tilde{U}_t' P_{B_t} \tilde{U}_t - \mathbb{E}\left[\tilde{U}_t' P_{B_t} \tilde{U}_t \right| z_t, \mathcal{F}_t \right] \right| \geq \delta \left| z_t, \mathcal{F}_t \right) \\ \leq \; & T \max_{1 \leq t \leq T} \mathbb{P}\left(J^{-d_z/2} \log(T)^{-1/2} \left| \tilde{U}_t' P_{B_t} \tilde{U}_t - \mathbb{E}\left[\tilde{U}_t' P_{B_t} \tilde{U}_t \right| z_t, \mathcal{F}_t \right] \right| \geq \delta \left| z_t, \mathcal{F}_t \right) \\ \leq \; & 2T \exp\left\{ -C \frac{\delta^2 \log(T)}{2K^4} \right\} \\ = \; & \exp\left\{ \log(2) + \log(T) \left[1 - C \frac{\delta^2}{2K^4} \right] \right\}, \end{aligned}$$

which can be made arbitrarily small for δ sufficiently large. Thus,

$$\max_{1 \leq t \leq T} \left| \tilde{U}_t' P_{B_t} \tilde{U}_t - \mathbb{E}\left[\tilde{U}_t' P_{B_t} \tilde{U}_t \right| z_t, \mathcal{F}_t \right] \right| / n_t = O_p\left(\log(T)^{1/2} J^{d_z/2} n^{-1} \right),$$

and $\max_{1 \leq t \leq T} |a' U_t' P_{B_t} U_t a| / n_t = O_p\left(J^{d_z} n^{-1} \right) + O_p\left(\log(T)^{1/2} J^{d_z/2} n^{-1} \right) = O_p\left(J^{d_z} n^{-1} \right)$. By similar steps we may show that equation (B.2) satisfies

$$\max_{1 \leq t \leq T} \left| a' U_t' U_t a - a' \mathbb{E}\left[U_t' U_t \right| \mathcal{F}_t \right] a \right| / n_t = O_p\left(\log(T)^{1/2} n^{-1/2} \right)$$

Finally, we need to deal with the term $U_t' M_{B_t} H_t / n_t$. First note that $\| U_t' M_{B_t} H_t / n_t \|^2 = n_t^{-2} \cdot tr\left(U_t' M_{B_t} H_t H_t' M_{B_t} U_t \right)$ so that we may focus on

$$\left| a' U_t' M_{B_t} H_t H_t' M_{B_t} U_t a \right| \leq \left| \tilde{U}_t' M_{B_t} H_t H_t' M_{B_t} \tilde{U}_t - \mathbb{E}\left[\tilde{U}_t' M_{B_t} H_t H_t' M_{B_t} \tilde{U}_t \right| z_t, \mathcal{F}_t \right] \right| + \mathbb{E}\left[\tilde{U}_t' M_{B_t} H_t H_t' M_{B_t} \tilde{U}_t \right| z_t, \mathcal{F}_t \right].$$

The second term has expectation

$$\begin{aligned} \max_{1 \leq t \leq T} \mathbb{E}\left[\tilde{U}_t' M_{B_t} H_t H_t' M_{B_t} \tilde{U}_t \right| z_t, \mathcal{F}_t \right] &= \max_{1 \leq t \leq T} \mathbb{E}\left[\mathrm{tr}\left(M_{B_t} H_t H_t' M_{B_t} \mathbb{E}\left[\tilde{U}_t \tilde{U}_t' \right| \mathcal{F}_t, z_t \right] \right) \right] \\ &\leq C \max_{1 \leq t \leq T} \mathbb{E}\left[\mathrm{tr}\left(\left(H_t - B_t \Pi_t^0 \right)' M_{B_t} \left(H_t - B_t \Pi_t^0 \right) \right) \right] \\ &\leq Cn \sum_{\ell=1}^{d_x} \max_{1 \leq t \leq T} \mathbb{E}\left[\max_{1 \leq j \leq J_t^{d_z}} \sup_{z} \left| \widehat{\mathbb{I}}_{jt}(z) h_{t,\ell}(z) - \widehat{\mathbb{I}}_{jt}(z) \pi_{jt,\ell}^0 \right|^2 \right] \\ &\leq CnJ^{-2}. \end{aligned}$$

Thus, by Markov's inequality it is $O_p\left(nJ^{-2} \right)$. For the first term consider we can again utilize the Hanson-Wright inequality which yields

$$\begin{aligned} & \mathbb{P}\left(\max_{1 \leq t \leq T} \left| \tilde{U}_t' M_{B_t} H_t H_t' M_{B_t} \tilde{U}_t - \mathbb{E}\left[\tilde{U}_t' M_{B_t} H_t H_t' M_{B_t} \tilde{U}_t \right| \mathcal{F}_t \right] \right| > \delta \left| z_t, \mathcal{F}_t \right) \\ \leq \; & 1 \wedge 2 \sum_{t=1}^{T} \exp\left\{ -C \min\left(\frac{\delta^2}{2K^4 \| M_{B_t} H_t H_t' M_{B_t} \|^2}, \frac{\delta}{K^2 \sup_{\|y\|=1} \| M_{B_t} H_t H_t' M_{B_t} y \|} \right) \right\} \end{aligned}$$

$$\leq \quad 1 \wedge 2T \max_{1\leq t\leq T} \exp\left\{-C\min\left(\frac{\delta^2}{2K^4\left\|M_{B_t}H_tH_t'M_{B_t}\right\|^2}, \frac{\delta}{K^2\left\|M_{B_t}H_tH_t'M_{B_t}\right\|}\right)\right\}$$

by properties of matrix norms. If we map $\delta \mapsto \delta \log(T)\, nJ^{-2}$ then

$$\begin{aligned}
& \mathbb{P}\left(\max_{1\leq t\leq T}\left|\tilde{U}_t'M_{B_t}H_tH_t'M_{B_t}\tilde{U}_t - \mathbb{E}\left[\tilde{U}_t'M_{B_t}H_tH_t'M_{B_t}\tilde{U}_t\right|\mathcal{F}_t\right]\right| > \delta\log(T)\, nJ^{-2}\right|\right) \\
\leq\ & \mathbb{E}\left[1\wedge 2\sum_{t=1}^{T}\exp\left\{-C\min\left(\frac{\delta^2\log(T)^2 n^2J^{-4}}{2K^4\left\|M_{B_t}H_tH_t'M_{B_t}\right\|^2}, \frac{\delta\log(T)\, nJ^{-2}}{K^2\left\|M_{B_t}H_tH_t'M_{B_t}\right\|}\right)\right\}\right] \\
\leq\ & C\cdot\mathbb{P}\left(\max_{1\leq t\leq T}\left\|M_{B_t}H_tH_t'M_{B_t}\right\| \geq C_2 nJ^{-2}\right) \\
& + \left(1\wedge 2\sum_{t=1}^{T}\exp\left\{-C\min\left(\frac{\delta^2\log(T)^2}{2K^4C_2^2}, \frac{\delta\log(T)}{K^2C_2}\right)\right\}\right)
\end{aligned}$$

The first term is $o(1)$ by Lemma 1 and the second term can be made arbitrarily small for sufficiently large δ so that

$$\max_{1\leq t\leq T}\left|\tilde{U}_t'M_{B_t}H_tH_t'M_{B_t}\tilde{U}_t - \mathbb{E}\left[\tilde{U}_t'M_{B_t}H_tH_t'M_{B_t}\tilde{U}_t\right|\mathcal{F}_t\right]\right| = O_p\left(\log(T)\, nJ^{-2}\right).$$

Thus,

$$\max_{1\leq t\leq T}\left\|U_t'M_{B_t}H_t/\, n_t\right\| = O_p\left(\log(T)\, n^{-1}J^{-2}\right) + O\left(n^{-1}J^{-2}\right) = O_p\left(\log(T)\, n^{-1}J^{-2}\right),$$

which is of smaller order than the term in equation (B.2). □

Proof of Lemma 5. We have

$$\begin{aligned}
\mathrm{V}(z) &= T^{-2}\sum_{t=1}^{T} n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\,\sigma_{it}^2 \\
&\leq CT^{-2}\sum_{t=1}^{T} n_t^{-2}\sum_{i=1}^{n_t}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it}) \\
&= CT^{-2}\sum_{t=1}^{T} n_t^{-1}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\,\hat{q}_{jt}^{-1} \\
&\leq CJ^d n^{-1}T^{-2}\sum_{t=1}^{T}\sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z) \\
&\leq CJ^d n^{-1}T^{-1} + CJ^d n^{-1}T^{-2}\sum_{t=1}^{T}\sum_{j=1}^{J_t^{d_z}} \left|\mathbf{1}_{jt}-1\right|\widehat{\mathbb{I}}_{jt}(z) \\
&\leq CJ^d n^{-1}T^{-1} + CJ^d n^{-1}T^{-1}\max_{1\leq t\leq T}\max_{1\leq j\leq J_t^{d_z}}\left|\mathbf{1}_{jt}-1\right|,
\end{aligned}$$

and so the second term is $o_p(1)$ by Lemma 2. The lower bound follows by similar steps.

□

Proof of Lemma 4. We have that

$$\mathbf{1}_{\beta,t}\left(\hat{\beta}_t - \beta_t\right) = \mathbf{1}_{\beta,t}\hat{\Omega}_{\mathrm{uu},t}^{-1}\, X_t'M_{B_t}\left(\mu(z_t)+\varepsilon_t\right)/\, n_t$$

Next recall that $X_t = U_t + H_t$ so we can decompose $1_{\beta,t}\left(\hat{\beta}_t - \beta_t\right)$ as

$$\begin{aligned} & T^{-1}\sum\nolimits_{t=1}^{T} \mathbf{1}_{\beta,t} s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1}\, X_t' M_{B_t}\left(\mu\left(z_t\right)+\varepsilon_t\right)/\, n_t \\ = \; & T^{-1}\sum\nolimits_{t=1}^{T} \mathbf{1}_{\beta,t} s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1}\, \left(U_t + H_t\right)' M_{B_t}\left(\mu\left(z_t\right)+\varepsilon_t\right)/\, n_t. \end{aligned}$$

For the first result it is then sufficient to consider $\sum_\ell \left|\mathcal{K}_{1\ell}\right|^2$ where

$$\begin{aligned} \mathcal{K}_{11} &= T^{-1}\sum\nolimits_{t=1}^{T} \mathbf{1}_{\beta,t} s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1}\, U_t'\varepsilon_t/\, n_t \\ \mathcal{K}_{12} &= T^{-1}\sum\nolimits_{t=1}^{T} \mathbf{1}_{\beta,t}\, s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1} U_t'\left(I_{n_t} - M_{B_t}\right)\varepsilon_t \Big/\, n_t \\ \mathcal{K}_{13} &= T^{-1}\sum\nolimits_{t=1}^{T} \mathbf{1}_{\beta,t}\, s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1} H_t' M_{B_t}\mu\left(z_t\right)\Big/\, n_t \\ \mathcal{K}_{14} &= T^{-1}\sum\nolimits_{t=1}^{T} \mathbf{1}_{\beta,t}\, s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1} H_t' M_{B_t}\varepsilon_t\Big/\, n_t \\ \mathcal{K}_{15} &= T^{-1}\sum\nolimits_{t=1}^{T} \mathbf{1}_{\beta,t}\, s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1} U_t' M_{B_t}\mu\left(z_t\right)\Big/\, n_t \end{aligned}$$

For the second result, by the CS inequality, it is sufficient to show that $\sum_\ell \mathcal{K}_{2\ell} = O_p\left(n^{-1}\right) + O_p\left(J^{-4}\right)$ where,

$$\begin{aligned} \mathcal{K}_{21} &= T^{-1}\sum\nolimits_{t=1}^{T}\left(\mathbf{1}_{\beta,t} s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1}\, U_t'\varepsilon_t/\, n_t\right)^2 \\ \mathcal{K}_{22} &= T^{-1}\sum\nolimits_{t=1}^{T}\left(\mathbf{1}_{\beta,t}\, s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1} U_t'\left(I_{n_t} - M_{B_t}\right)\varepsilon_t\Big/\, n_t\right)^2 \\ \mathcal{K}_{23} &= T^{-1}\sum\nolimits_{t=1}^{T}\left(\mathbf{1}_{\beta,t}\, s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1} H_t' M_{B_t}\mu\left(z_t\right)\Big/\, n_t\right)^2 \\ \mathcal{K}_{24} &= T^{-1}\sum\nolimits_{t=1}^{T}\left(\mathbf{1}_{\beta,t}\, s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1} H_t' M_{B_t}\varepsilon_t\Big/\, n_t\right)^2 \\ \mathcal{K}_{25} &= T^{-1}\sum\nolimits_{t=1}^{T}\left(\mathbf{1}_{\beta,t}\, s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1} U_t' M_{B_t}\mu\left(z_t\right)\Big/\, n_t\right)^2 \end{aligned}$$

We will prove the second result, first. Consider $\mathcal{K}_{21}$

$$\begin{aligned} \left|\mathcal{K}_{21}\right| &= T^{-1}\sum\nolimits_{t=1}^{T}\left|\mathbf{1}_{\beta,t} s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1}\, U_t'\varepsilon_t/\, n_t\right|^2 \\ &\le T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}\lambda_{\max}\left(\hat{\Omega}_{\mathrm{uu},t}^{-1}\right)^2 \left\|s_t\right\|^2 \left\|U_t'\varepsilon_t/\, n_t\right\|^2 \\ &\le CT^{-1}\sum\nolimits_{t=1}^{T} n_t^{-2}\mathrm{tr}\left(U_t'\varepsilon_t\varepsilon_t' U_t\right). \end{aligned}$$

Taking expectations we obtain,

$$T^{-1}\sum\nolimits_{t=1}^{T} n_t^{-2}\mathbb{E}\left[\mathrm{tr}\left(U_t' E\left[\varepsilon_t\varepsilon_t'\middle|\, z_t, x_t, \mathcal{F}_t\right] U_t\right)\right] \le Cn^{-1}.$$

Thus, $\mathcal{K}_{21} = O_p\left(n^{-1}\right)$ by Markov's inequality. By similar steps we can show that $\mathcal{K}_{22} = O_p\left(J^{d_z} n^{-2}\right)$. Next consider $\mathcal{K}_{23}$

$$\begin{aligned} \mathcal{K}_{23} &= T^{-1}\sum\nolimits_{t=1}^{T} n_t^{-2}\left(\mathbf{1}_{\beta,t} s_t' \hat{\Omega}_{\mathrm{uu},t}^{-1} H_t' M_{B_t}\mu\left(z_t\right)\mu\left(z_t\right)' M_{B_t} H_t \hat{\Omega}_{\mathrm{uu},t}^{-1} s_t\right) \\ &\le CT^{-1}\sum\nolimits_{t=1}^{T} n_t^{-2}\left(\mathbf{1}_{\beta,t}\left[\lambda_{\max}\left(\hat{\Omega}_{\mathrm{uu},t}^{-1}\right)\right]^2\left\|M_{B_t}\left(H_t - B_t\Pi_t^0\right)\right\|^2\left\|M_{B_t}\left(\mu\left(z_t\right) - B_t\gamma_t^0\right)\right\|^2\right) \\ &\le CT^{-1}\sum\nolimits_{t=1}^{T} n_t^{-2}\left(\sum\nolimits_{i_1=1}^{n_t}\left\|h_t\left(z_{i_1 t}\right) - B_t\left(z_{i_1 t}\right)'\pi_t^0\right\|^2\right)\left(\sum\nolimits_{i_2=1}^{n_t}\left\|\mu\left(z_{i_2 t}\right) - B_t\left(z_{i_2 t}\right)'\gamma_t^0\right\|^2\right) \\ &\le CT^{-1}\sum\nolimits_{t=1}^{T} n_t^{-2}\left(\sum\nolimits_{i_1=1}^{n_t}\sum\nolimits_{\ell=1}^{d_x}\sum\nolimits_{j_1=1}^{J_t^{d_z}}\left|\widehat{\mathbb{I}}_{j_1 t}\left(z_{it}\right)h_{t,\ell}\left(z_{it}\right) - \widehat{\mathbb{I}}_{j_1 t}\left(z_{it}\right)\pi_{j_1 t,\ell}^0\right|^2\right)\times \end{aligned}$$

$$\left(\sum\nolimits_{i_2=1}^{n_t}\sum\nolimits_{j_2=1}^{J_t^{d_z}}\left|\widehat{\mathbb{I}}_{j_2t}\left(z_{it}\right)\mu\left(z_{it}\right)-\widehat{\mathbb{I}}_{j_2t}\left(z_{it}\right)\gamma_{j_1t}^0\right|^2\right).$$

Thus, $\mathcal{K}_{23}=O_p\left(J^{-4}\right)$. Now consider $\mathcal{K}_{24}$

$$\begin{aligned}\mathcal{K}_{24} &= T^{-1}\sum\nolimits_{t=1}^{T}\left|\mathbf{1}_{\beta,t}\; s_t'\hat{\Omega}_{\mathrm{uu},t}^{-1}H_t'M_{B_t}\varepsilon_t\big/\, n_t\right|^2\\ &\leq CT^{-1}\sum\nolimits_{t=1}^{T}\left\|H_t'M_{B_t}\varepsilon_t\big/\, n_t\right\|^2.\end{aligned}$$

Taking expectations gives

$$\begin{aligned}& T^{-1}\sum\nolimits_{t=1}^{T}n_t^{-2}\mathbb{E}\left[\mathrm{tr}\left(H_t'M_{B_t}\mathbb{E}\left[\varepsilon_t\varepsilon_t'\right|z_t,x_t,\mathcal{F}_t\right]M_{B_t}H_t\right)\right]\\ \leq\;& CT^{-1}\sum\nolimits_{t=1}^{T}n_t^{-2}\mathbb{E}\left[\mathrm{tr}\left(\left(H_t-B_t\Pi_t^0\right)'\left(H_t-B_t\Pi_t^0\right)\right)\right]\\ \leq\;& CT^{-1}\sum\nolimits_{t=1}^{T}n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{\ell=1}^{d_x}\mathbb{E}\left[\max_{1\leq j\leq J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}\left(z\right)h_{t,\ell}\left(z\right)-\widehat{\mathbb{I}}_{jt}\left(z\right)\pi_{jt,\ell}^0\right|^2\sum\nolimits_{j=1}^{J_t^{d_z}}\widehat{\mathbb{I}}_{jt}\left(z\right)\right]\\ \leq\;& Cn^{-1}J^{-2}.\end{aligned}$$

Thus $\mathcal{K}_{24}=O_p\left(n^{-1}J^{-2}\right)$ by Markov's inequality. Now consider $\mathcal{K}_{25}$

$$\begin{aligned}\mathcal{K}_{25} &= T^{-1}\sum\nolimits_{t=1}^{T}\left|\mathbf{1}_{\beta,t}\; s_t'\hat{\Omega}_{\mathrm{uu},t}^{-1}U_t'M_{B_t}\mu\left(z_t\right)\big/\, n_t\right|^2\\ &\leq CT^{-1}\sum\nolimits_{t=1}^{T}\left\|U_t'M_{B_t}\mu\left(z_t\right)\big/\, n_t\right\|^2.\end{aligned}$$

Taking expectations gives

$$\begin{aligned}& T^{-1}\sum\nolimits_{t=1}^{T}n_t^{-2}\mathbb{E}\left[\left\|U_t'M_{B_t}\mu\left(z_t\right)\right\|^2\right]\\ \leq\;& CT^{-1}\sum\nolimits_{t=1}^{T}n_t^{-2}\mathbb{E}\left[\mathrm{tr}\left(\mathbb{E}\left[U_tU_t'\right|z_t,\mathcal{F}_t\right]M_{B_t}\mu\left(z_t\right)\mu\left(z_t\right)'M_{B_t}\right)\right]\\ \leq\;& CT^{-1}\sum\nolimits_{t=1}^{T}n_t^{-2}\mathbb{E}\left[\mathrm{tr}\left(\left(\mu\left(z_t\right)-B_t\gamma_t^0\right)'\left(\mu\left(z_t\right)-B_t\gamma_t^0\right)\right)\right]\\ \leq\;& CT^{-1}\sum\nolimits_{t=1}^{T}n_t^{-2}\sum\nolimits_{i=1}^{n_t}\mathbb{E}\left[\max_{1\leq j\leq J_t^{d_z}}\sup_z\left|\widehat{\mathbb{I}}_{jt}\left(z\right)\mu\left(z\right)-\widehat{\mathbb{I}}_{jt}\left(z\right)\gamma_t^0\right|^2\right]\\ \leq\;& Cn^{-1}J^{-2}.\end{aligned}$$

Thus, $\mathcal{K}_{25}=O_p\left(n^{-1}J^{-2}\right)$ and

$$\sum_{\ell}\mathcal{K}_{2\ell}=O_p\left(n^{-1}\right)+O_p\left(J^{d_z}n^{-2}\right)+O_p\left(J^{-4}\right)+O_p\left(n^{-1}J^{-2}\right)=O_p\left(n^{-1}\right)+O_p\left(J^{-4}\right),$$

where the second equality follows by Assumption 3. Now consider the first result. We have that $\mathcal{K}_{11}$ satisfies $\mathcal{K}_{11}=\mathcal{K}_{111}+\mathcal{K}_{112}$ where

$$\begin{aligned}\mathcal{K}_{111} &= T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}s_t'\Omega_{\mathrm{uu},t}^{-1}\;U_t'\varepsilon_t\big/\, n_t\\ \mathcal{K}_{112} &= T^{-1}\sum\nolimits_{t=1}^{T}\mathbf{1}_{\beta,t}s_t'\Omega_{\mathrm{uu},t}^{-1}\left(\Omega_{\mathrm{uu},t}-\hat{\Omega}_{\mathrm{uu},t}\right)\hat{\Omega}_{\mathrm{uu},t}^{-1}\;U_t'\varepsilon_t\big/\, n_t.\end{aligned}$$

For $\mathcal{K}_{111}$ we have

$$\begin{aligned}\mathbb{E}\left|\mathcal{K}_{11}\right|^2 &= T^{-2}\sum\nolimits_{t_1,t_2}n_{t_1}^{-1}n_{t_2}^{-1}\mathbb{E}\left[\mathbf{1}_{\beta,t_1}\mathbf{1}_{\beta,t_2}s_{t_1}'\Omega_{\mathrm{uu},t_1}^{-1}U_{t_1}'\mathbb{E}\left[\varepsilon_{t_1}\varepsilon_{t_2}'\right|\mathcal{F}_{t_1},\mathcal{F}_{t_2},z_{t_1},z_{t_2},x_{t_1},x_{t_2}\right]U_{t_2}\Omega_{\mathrm{uu},t_2}^{-1}s_{t_2}\right]\\ &= T^{-2}\sum\nolimits_{t_1}n_{t_1}^{-2}\mathbb{E}\left[\mathbf{1}_{\beta,t_1}s_{t_1}'\Omega_{\mathrm{uu},t_1}^{-1}U_{t_1}'\varepsilon_{t_1}\varepsilon_{t_1}'U_{t_1}\Omega_{\mathrm{uu},t_1}^{-1}s_{t_1}\right]\\ &\leq Cn^{-1}T^{-1},\end{aligned}$$

following similar steps as for the term $\mathcal{K}_{21}$. Then, by the CS inequality

$$\begin{aligned}
|\mathcal{K}_{112}|^2 &\leq T^{-1}\sum_{t=1}^{T} \mathbf{1}_{\beta,t} \left\| s_t' \Omega_{\mathrm{uu},t}^{-1} \left(\Omega_{\mathrm{uu},t} - \hat{\Omega}_{\mathrm{uu},t} \right) \hat{\Omega}_{\mathrm{uu},t}^{-1} \right\|^2 \times T^{-1}\sum_{t=1}^{T} \left\| U_t' \varepsilon_t / n_t \right\|^2 \\
&\leq T^{-1}\sum_{t=1}^{T} \mathbf{1}_{\beta,t} \left[\lambda_{\max} \left(\Omega_{\mathrm{uu},t}^{-1} \right) \right]^2 \left[\lambda_{\max} \left(\hat{\Omega}_{\mathrm{uu},t}^{-1} \right) \right]^2 \left\| \hat{\Omega}_{\mathrm{uu},t} - \Omega_{\mathrm{uu},t} \right\|^2 \times T^{-1}\sum_{t=1}^{T} \left\| U_t' \varepsilon_t / n_t \right\|^2 \\
&\leq CT^{-1}\sum_{t=1}^{T} \left\| \hat{\Omega}_{\mathrm{uu},t} - \Omega_{\mathrm{uu},t} \right\|^2 \times T^{-1}\sum_{t=1}^{T} \left\| U_t' \varepsilon_t / n_t \right\|^2
\end{aligned}$$

The first factor is $O_p\left(n^{-1}\right) + O_p\left(J^{-4}\right) + O_p\left(J^{2d_z} n^{-2}\right)$ by Lemma 3 and by similar steps as for $\mathcal{K}_{21}$ the second factor has expectation,

$$T^{-1}\sum_{t=1}^{T} \mathbb{E}\left[\left\| U_t' \varepsilon_t / n_t \right\|^2 \right] \leq C n^{-1}.$$

Thus, $|\mathcal{K}_{11}|^2 = O_p\left(n^{-1}T^{-1}\right) + O_p\left(n^{-2}\right) + O_p\left(n^{-1}J^{-4}\right) + O_p\left(J^{2d_z} n^{-3}\right)$ by Markov's inequality. By similar steps we can show that

$$\begin{aligned}
|\mathcal{K}_{12}|^2 &= O_p\left(T^{-1}n^{-2}J^{d_z}\right) + O_p\left(J^{d_z}n^{-3}\right) + O_p\left(J^{d_z-4}n^{-2}\right) + O_p\left(J^{3d_z}n^{-4}\right) \\
\mathcal{K}_{13} &= O_p\left(J^{-2}\right) \\
|\mathcal{K}_{14}|^2 &= O_p\left(n^{-1}J^{-2}\right) + O_p\left(n^{-2}J^{-2}\right) + O_p\left(n^{-1}J^{-6}\right) + O_p\left(J^{2d_z-2}n^{-3}\right) \\
|\mathcal{K}_{15}|^2 &= O_p\left(n^{-1}T^{-1}J^{-2}\right) + O_p\left(n^{-2}J^{-2}\right) + O_p\left(n^{-1}J^{-6}\right) + O_p\left(J^{2d_z-2}n^{-3}\right).
\end{aligned}$$

Thus, we have that

$$\sum_{\ell} |\mathcal{K}_{1\ell}|^2 = O_p\left(n^{-1}T^{-1}\right) + O_p\left(J^{-4}\right) + O_p\left(J^{2d_z}n^{-3}\right) + O_p\left(J^{d_z-4}n^{-2}\right).$$

□

Proof of Lemma 6. We would like to show that

$$\mathrm{V}(z)^{-1} T^{-2} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z) \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it}) \left(\varepsilon_{it}^2 - \sigma_{it}^2 \right) = o_p(1).$$

By Lemma 5 we need only show that

$$\begin{aligned}
& J^{-d} n T^{-1} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z) \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it}) \left(\varepsilon_{it}^2 - \sigma_{it}^2 \right) \\
= & J^{-d} n T^{-1} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z) \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it}) \eta_{it}^{+} \\
& + J^{-d} n T^{-1} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z) \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it}) \eta_{it}^{-} \\
= & \mathcal{J}_1 + \mathcal{J}_2,
\end{aligned}$$

where

$$\begin{aligned}
\eta_{it}^{+} &= \varepsilon_{it}^2 \mathbf{1}\left\{ |\varepsilon_{it}| > t_{nT} \right\} - \mathbb{E}\left[\varepsilon_{it}^2 \mathbf{1}\left\{ |\varepsilon_{it}| > t_{nT} \right\} \middle| \mathcal{F}_t, z_{it}, x_{it} \right] \\
\eta_{it}^{-} &= \varepsilon_{it}^2 \mathbf{1}\left\{ |\varepsilon_{it}| \leq t_{nT} \right\} - \mathbb{E}\left[\varepsilon_{it}^2 \mathbf{1}\left\{ |\varepsilon_{it}| \leq t_{nT} \right\} \middle| \mathcal{F}_t, z_{it}, x_{it} \right]
\end{aligned}$$

First consider $\mathcal{J}_1$,

$$\begin{aligned}
& \mathbb{P}\left(\left| J^{-d} n T^{-1} \sum_{t=1}^{T} n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z) \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it}) \eta_{it}^{+} \right| > \delta_{nT} \right) \\
\leq & \frac{n^2}{\delta_{nT}^2 J^{2d} T^2} \times
\end{aligned}$$

$$
\begin{aligned}
& \sum\nolimits_{t_1,t_2=1}^{T} n_{t_1}^{-2} n_{t_2}^{-2} \sum\nolimits_{i_1,i_2=1}^{n_t} \sum\nolimits_{j_1,j_2=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbf{1}_{j_1t_1}\mathbf{1}_{j_2t_2}\widehat{\mathbb{I}}_{j_1t_1}(z)\widehat{\mathbb{I}}_{j_2t_2}(z)\hat{q}_{j_1t_1}^{-2}\hat{q}_{j_2t_2}^{-2}\widehat{\mathbb{I}}_{j_1t_1}(z_{i_1t_1})\widehat{\mathbb{I}}_{j_2t_2}(z_{i_2t_2})\eta_{i_1t_1}^{+}\eta_{i_2t_2}^{+}\right] \\
= \;& \frac{n^2}{\delta_{nT}^2 J^{2d} T^2} \times \sum\nolimits_{t=1}^{T} n_t^{-4} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-4}\widehat{\mathbb{I}}_{jt}(z_{it})\left(\eta_{it}^{+}\right)^2\right] \\
\leq \;& \frac{n^2}{\delta_{nT}^2 J^{2d} T^2} \sum\nolimits_{t=1}^{T} n_t^{-4} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-4}\widehat{\mathbb{I}}_{jt}(z_{it})\,\mathbb{E}\left[\varepsilon_{it}^4\mathbf{1}\left\{|\varepsilon_{it}|>t_{nT}\right\}\middle|\mathcal{F}_t, z_{it}, x_{it}\right]\right] \\
\leq \;& \frac{n^2}{\delta_{nT}^2 t_{nT}^{\varrho} J^{2d} T^2} \sum\nolimits_{t=1}^{T} n_t^{-4} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-4}\widehat{\mathbb{I}}_{jt}(z_{it})\,\mathbb{E}\left[|\varepsilon_{it}|^{4+\varrho}\middle|\mathcal{F}_t, z_{it}, x_{it}\right]\right] \\
\leq \;& \frac{Cn^2}{\delta_{nT}^2 t_{nT}^{\varrho} J^{2d} T^2} \sum\nolimits_{t=1}^{T} n_t^{-4} \sum\nolimits_{i=1}^{n_t} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-4}\widehat{\mathbb{I}}_{jt}(z_{it})\right] \\
\leq \;& \frac{CJ^{2d}n^2}{\delta_{nT}^2 t_{nT}^{\varrho} T^2} \sum\nolimits_{t=1}^{T} n_t^{-3} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\, n_t^{-1}\sum\nolimits_{i=1}^{n_t}\widehat{\mathbb{I}}_{jt}(z_{it})\right] \\
\leq \;& \frac{CJ^{d}n^2}{\delta_{nT}^2 t_{nT}^{\varrho} T^2} \sum\nolimits_{t=1}^{T} n_t^{-3} \sum\nolimits_{j=1}^{J_t^{d_z}} \mathbb{E}\left[\mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\right] \\
\leq \;& \frac{CJ^{d}}{\delta_{nT}^2 t_{nT}^{\varrho} nT}.
\end{aligned}
$$

Now consider $\mathcal{J}_2$:

$$
\begin{aligned}
& \left|J^{-d}nT^{-1}\sum\nolimits_{t=1}^{T} n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\eta_{it}^{-}\right| \\
\leq \;& J^{-d}nT^{-1}\sum\nolimits_{t=1}^{T}\left|n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\eta_{it}^{-}\right| \\
\leq \;& J^{-d}n\max_{1\leq t\leq T}\left|n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\eta_{it}^{-}\right|,
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{P}\left(J^{-d}n\max_{1\leq t\leq T}\left|n_t^{-2}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\eta_{it}^{-}\right|>\delta_{nT}\right) \\
\leq \;& \sum\nolimits_{t=1}^{T}\mathbb{P}\left(\left|n_t^{-1}\sum\nolimits_{i=1}^{n_t} J^{-d}nn_t^{-1}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\eta_{it}^{-}\right|>\delta_{nT}\right).
\end{aligned}
$$

This is a mean-zero, bounded random variable. The summands are bounded by

$$
\begin{aligned}
& \left|J^{-d}nn_t^{-1}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\eta_{it}^{-}\right| \\
\leq \;& J^{-d}nn_t^{-1}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\left|\eta_{it}^{-}\right| \\
\leq \;& t_{nT}^2 J^d nn_t^{-1}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\widehat{\mathbb{I}}_{jt}(z_{it}) \\
\leq \;& Ct_{nT}^2 J^d.
\end{aligned}
$$

The rescaled sum of the variances are

$$
\begin{aligned}
& \frac{1}{n_t}\sum\nolimits_{i=1}^{n_t}\mathbb{E}\left[\left|J^{-d}nn_t^{-1}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-2}\widehat{\mathbb{I}}_{jt}(z_{it})\eta_{it}^{-}\right|^2\middle|\mathcal{F}_t, z_t, x_t\right] \\
\leq \;& \frac{C}{n_tJ^{2d}}\sum\nolimits_{i=1}^{n_t}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-4}\widehat{\mathbb{I}}_{jt}(z_{it})\,\mathbb{E}\left[\left|\eta_{it}^{-}\right|^2\middle|\mathcal{F}_t, z_t, x_t\right] \\
\leq \;& \frac{C}{J^{2d}}\sum\nolimits_{j=1}^{J_t^{d_z}} \mathbf{1}_{jt}\widehat{\mathbb{I}}_{jt}(z)\hat{q}_{jt}^{-4}\left(n_t^{-1}\sum\nolimits_{i=1}^{n_t}\widehat{\mathbb{I}}_{jt}(z_{it})\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad CJ^d \sum_{j=1}^{J_t^{dz}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z) \\
&\leq \quad CJ^d.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \mathbb{P}\left(J^{-d} n \max_{1\leq t\leq T} \left| n_t^{-2} \sum_{i=1}^{n_t} \sum_{j=1}^{J_t^{dz}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z)\, \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it})\, \eta_{it}^{-} \right| > \delta_{nT} \right) \\
\leq \quad & 1 \wedge \sum_{t=1}^{T} \mathbb{P}\left(\left| n_t^{-1} \sum_{i=1}^{n_t} J^{-d} n n_t^{-1} \sum_{j=1}^{J_t^{dz}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z)\, \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it})\, \eta_{it}^{-} \right| > \delta_{nT} \right) \\
\leq \quad & \mathbb{E}\left[1 \wedge \sum_{t=1}^{T} \mathbb{P}\left(\left| n_t^{-1} \sum_{i=1}^{n_t} J^{-d} n n_t^{-1} \sum_{j=1}^{J_t^{dz}} \mathbf{1}_{jt} \widehat{\mathbb{I}}_{jt}(z)\, \hat{q}_{jt}^{-2} \widehat{\mathbb{I}}_{jt}(z_{it})\, \eta_{it}^{-} \right| > \delta_{nT} \middle| \mathcal{F}_t, z_t, x_t \right) \right] \\
\leq \quad & \mathbb{E}\left[1 \wedge C \sum_{t=1}^{T} \exp\left\{ -\frac{n_t \delta_{nT}^2}{C_1 J^d + C_2 \delta_{nT} J^d t_{nT}^2} \right\} \right] \\
\leq \quad & 1 \wedge CT \exp\left\{ -\frac{C n \delta_{nT}^2 J^{-d}}{1 + \delta_{nT} t_{nT}^2} \right\} \\
= \quad & 1 \wedge C \exp\left\{ \log(T) \left(1 - \frac{C n \delta_{nT}^2 J^{-d} \log(T)^{-1}}{1 + \delta_{nT} t_{nT}^2} \right) \right\}.
\end{aligned}
$$

Thus, we need to find conditions such that

$$
\begin{aligned}
t_{nT} &\to \infty \\
\delta_{nT} &\to 0 \\
\frac{J^d}{\delta_{nT}^2 t_{nT}^{\varrho} nT} &= O(1) \\
\frac{J^d \log(T)}{n \delta_{nT}^2} &\not\to \infty \\
\frac{J^d \log(T)\, t_{nT}^2}{n \delta_{nT}} &\not\to \infty.
\end{aligned}
$$

Let $t_{nT} = \log\left(T \vee J^d\right)^{1/4} \log\left(T \wedge J^d\right)^{-1/4}$ and $\delta_{nT}^2 = J^d n^{-1} \log\left(T \vee J^d\right)$. Then, in reverse order, we have

$$
\frac{J^d \log(T)\, t_{nT}^2}{n \delta_{nT}} = \frac{J^{d/2} \log(T) \log\left(T \wedge J^d\right)^{-1/2}}{n^{1/2}} = \sqrt{\frac{J^d \log(T)^2 \log\left(T \wedge J^d\right)^{-1}}{n}},
$$

which is $O(1)$ by assumption. Then

$$
\frac{J^d \log(T)}{n \delta_{nT}^2} = \frac{\log(T)}{\log\left(T \vee J^d\right)} = O(1),
$$

and

$$
\frac{J^d}{\delta_{nT}^2 t_{nT}^{\varrho} nT} = \frac{1}{\log\left(T \vee J^d\right) \log\left(T \vee J^d\right)^{\varrho/4} \log\left(T \wedge J^d\right)^{-\varrho/4} T} = o(1).
$$

□

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